QUARTERLY OF APPLIED MATHEMATICS

Vol. IX October, 1951 No. 3

ON A QUASI-LINEAR PARABOLIC EQUATION OCCURRING IN AERODYNAMICS*

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1. Introduction. The equation under discussion in this paper is the following:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $u = u(x, t)$ in some domain and ν is a parameter. The occurrence of the first derivative in t and the second in x clearly indicates the equation is parabolic, similar to the heat equation, while the interesting additional feature is the occurrence of the non-linear term $u\,\partial u/\partial x$. The equation thus shows a structure roughly similar to that of the Navier-Stokes equations and has actually appeared in two separate problems in aerodynamics. An equation simply related to (1) appears in the approximate theory of a weak non-stationary shock wave in a real fluid. This is discussed in Ref. 1 (pp. 146-154) where a general solution of (1) is given. The equation is also given in J. Burgers' theory of a model of turbulence (Ref. 2) where he notes the relationship between the model theory and the shock wave. Historically, the equation (1) first appears in a paper by H. Bateman (Ref. 3) in 1915 when he mentioned it as worthy of study and gave a special solution. Eq. (1) is of some mathematical interest in itself and may have applications in the theory of stochastic processes. The aim of this paper is to study the general properties of (1) and relate the various applications. I wish to thank Professor P. A. Lagerstrom and F. K. Chuang for helpful collaboration.

2. Relationship of (1) to Shock Wave Theory. The solutions to Eq. (1) can approximately describe the flow through a shock wave in a viscous fluid. They can be related to the shock wave in several ways. In Ref. 1 an approximation based on the Navier-Stokes equations for one-dimensional non-stationary flow of a compressible viscous fluid gives

$$\frac{\partial w}{\partial t} + \beta w\frac{\partial w}{\partial x} = \frac{4}{3}\nu^*\frac{\partial^2 w}{\partial x^2} \tag{2}$$

for w = excess of flow velocity over a sonic velocity where $\beta = (\gamma + 1)/2$, ν^* = kinematic viscosity at sonic conditions. Eq. (2) is reduced to Eq. (1) by $\beta w = u$; $4/3\ \nu^* = \nu$. In this paper a different discussion, intended to illustrate the production and maintenance of a shock, will be given. In the cases in which we are interested we can say that

*Received Apr. 17, 1950. This work is the result of research carried out under ONR contract N6-onr-244 Task Order VIII.

ν is a small parameter, a statement which will be made more precise later. Thus, we study the case $\nu = 0$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{3}$$

in order to see the underlying mechanism of propagation. Eq. (3) is similar to the non-linear equations for propagation of waves of finite amplitude in one dimension (Ref. 4, p. 482)

$$\begin{aligned} \left\{\frac{\partial}{\partial t} + (u + c)\frac{\partial}{\partial x}\right\}(\omega + u) &= 0, \\ \left\{\frac{\partial}{\partial t} + (u - c)\frac{\partial}{\partial x}\right\}(\omega - u) &= 0, \end{aligned} \tag{4}$$

where $\omega = \int_{\rho_0}^{\rho} (dp/d\rho)^{1/2}\, d\rho/\rho$; $c = (dp/d\rho)^{1/2}$ = velocity of sound
p = pressure, ρ = density.

Eq. (4) reduces to (3) exactly if $c = 0$. Thus we have as a model a fluid in which u is transported by the fluid motion itself (i.e. with a velocity u). According to the theory of characteristics for first order equations, the projection of the characteristics of (3) on the (x, t) plane are straight lines whose slope is

$$\frac{dx}{dt} = u. \tag{5}$$

In addition Eq. (3) interpreted geometrically states that u is constant in the characteristic direction. Thus u corresponds to the Riemann invariants $u \pm \omega$ of (4). It follows that each characteristic is straight over its length and carries a definite value u. Therefore knowing the values of u at any particular instant the solution for the future may be found by following the characteristics. However this process may terminate after a finite time when two characteristics intersect. For example this must happen if for $x_1 < x_2$, $u(x_1, t) > u(x_2, t)$. This phenomenon is the steepening of the wave front for waves of finite amplitude known from the study of (4).

We can now regard the viscosity as a mechanism for preventing the formation of discontinuities. The viscous stresses depend on changes of rates of strain so that the viscosity ν appears in Eq. (1) multiplying a term of higher order $\partial^2 u/\partial x^2$. This gives (1) the nature of a diffusion equation where velocity (actually momentum) is the quantity diffused. It is, of course, very much different from solid friction as expressed in first order damping terms. The characteristics of (1) are different from those of (3) for any $\nu > 0$ and they are given in the (x, t) plane by the curves $\psi(x, t)$ = const. where

$$\nu\psi_x^2 = 0 \qquad \text{or } t = \text{const.} \tag{6}$$

These characteristics occur as a double set which can be regarded as the limit of a pair of characteristics indicating a high signal speed. Thus here the speed of signals is infinite.

Hence Eq. (1) shows the typical features of shock wave theory: (i) A non-linear term tending to steepen the wave fronts and produce complete dissipation, (ii) A viscous term of higher order which prevents formation of actual discontinuities and which tends to diffuse any differences in velocity.

3. Relationship of (1) to Turbulence Theory. Eq. (1) is related to turbulence theory as a mathematical model. The similarity of the Navier-Stokes equation to Eq. (1) is

responsible. Both contain non-linear terms of the type: unknown function times a first derivative; and both contain higher order terms multiplied by a small parameter. The problems in turbulence are not very well defined but in most theories one is interested in some kind of spectrum, the feeding of energy through the spectrum and dissipation of energy. The model equation contains the non-linear terms and viscous terms vital to a study of those topics. Following Burgers, we regard Eq. (1) as a model for decaying free turbulence; he discusses other cases in much detail.

The mathematical (rather than physical) aspect of the model can be emphasized as follows. We study as before the underlying wave propagation for $\nu = 0$. For the usual turbulence theory we are dealing with flows in two or three dimensions so that in addition to equations like (1) a kinematical or continuity restriction must be added. For an incompressible fluid in two dimensions we have

$$u_t + uu_x + vu_y = -\frac{1}{\rho}p_x, \tag{7a}$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho}p_y, \tag{7b}$$

$$u_x + v_y = 0. \tag{7c}$$

The continuity equation (7c) is a statement that only transversal waves are present in the flow field. However as discussed in Sec. 2 a typical feature of (1) is the longitudinal steepening effect (i.e. steepening in the flow direction). It does not seem possible to have waves of that type for a system like (7) but the steepening must be transverse to the local flow direction. The underlying structure of (7) is given by the characteristic surfaces $\Psi(x, y, t) = $ const. which now satisfy the equation

$$\frac{1}{\rho}(\Psi_x^2 + \Psi_y^2)(\Psi_t + u\Psi_x + v\Psi_y) = 0. \tag{8}$$

The vanishing of the first factor gives a double set of surfaces $t = $ const. corresponding to the characteristic cones which have degenerated into planes, and propagation of pressure signals with infinite speed. The vanishing of the second factor gives stream surfaces. If any steepening effect occurs it will have to be related to these surfaces. In this case an invariant on these stream surfaces is the vorticity $\xi(x, y, t)$ which satisfies the equation

$$\xi_t + u\xi_x + v\xi_y = 0. \tag{9}$$

Thus, as pointed out by Burgers, the situation in the actual case is complicated very much by the kinematical restrictions.

The quantity u in the above model is, of course, a measure of the turbulence. The model is completed by some kind of statistical analysis based on Eq. (1) or else Eq. (1) is considered as a stochastic differential equation subject to random boundary values.

4. General Properties of Eq. (1). It may be expected that Eq. (1) is similar to the heat equation

$$\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2} \tag{10a}$$

and to the heat equation in a moving medium

$$\frac{\partial \theta}{\partial t} + U \frac{\partial \theta}{\partial x} = \nu \frac{\partial^2 \theta}{\partial x^2} \tag{10b}$$

with respect to the type of boundary value problems which are sensible. In this section some comparisons will be made and other general properties of (1) will be studied.

An energy equation for (1) can be found by multiplying (1) by u and integrating over a spatial domain, for example $(x_1 \leq x \leq x_2)$. This gives

$$\begin{aligned} &\frac{1}{2} \int_{x_1}^{x_2} \frac{\partial}{\partial t} (u^2)\, dx + \frac{1}{3} \{u^3(x_2, t) - u^3(x_1, t)\} \\ &\quad = \nu \left\{ u(x_2, t) \frac{\partial u}{\partial x}(x_2, t) - u(x_1, t) \frac{\partial u}{\partial x}(x_1, t) \right\} - \nu \int_{x_1}^{x_2} \left(\frac{\partial u}{\partial x}\right)^2 dx. \end{aligned} \tag{11}$$

The various terms in Eq. (11) have the following meaning:

$\frac{1}{2} \int_{x_1}^{x_2} \frac{\partial u^2}{\partial t}\, dx$ = total rate of change of kinetic energy in system,

$\frac{1}{3} \{u^3(x_2, t) - u^3(x_1, t)\}$ = net flux of kinetic energy out of system across boundaries,

$\left\{ \left(u \frac{\partial u}{\partial x}\right)_{x_2} - \left(u \frac{\partial u}{\partial x}\right)_{x_1} \right\}$ = rate of work done on system at boundaries,

$\nu \int_{x_1}^{x_2} \left(\frac{\partial u}{\partial x}\right)^2 dx$ = total dissipation of energy by viscosity in system.

The non-linear term in (1) provides a means of feeding energy into the system across the the boundaries. We can have a steady-state solution to (1) in an infinite domain $(-\infty < x < \infty)$ with an energy balance.

$$\frac{1}{3} (u_1^3 - u_2^3) = \nu \int_{-\infty}^{+\infty} \left(\frac{\partial u}{\partial x}\right)^2 dx, \tag{12}$$

where $u_1 = u(-\infty, t)$, $u_2 = u(+\infty, t)$ and $u_1 > u_2$. A linear equation like (10a) or (10b) can have no such (bounded, non-zero) solution in an infinite domain. For such a steady state the total dissipation, as given by (12), is independent of the value of ν. The steady solution is actually

$$u = -u_1 \tanh \frac{u_1(x - x_1)}{2\nu}, \tag{13}$$

where u_1 and x_1 are constants. Thus $u_2 = -u_1$. This gives the steady flow through a shock wave and shows how the non-linear terms are responsible for a change from $u > 0$ (supersonic) to $u < 0$ (subsonic). As $\nu \to 0$ the steep front of (13) at $x = x_1$ approaches a discontinuity which corresponds to the shock wave in a fluid where $\nu = 0$. This shows one reason why the conditions on continuity of solutions for $\nu = 0$ have to be relaxed if the solutions are to correspond to reality. Using (13) it is also possible to give steady state solutions for finite domains $(0 < x < l)$ which have regions of rapid transition either in the interior or adjacent to the boundary.

A translation property of (1) is also of interest. If we consider (1) in a coordinate system moving in the positive x-direction with a constant velocity U, defined by

$$\begin{aligned} \bar{x} &= x - Ut \\ \bar{t} &= t \end{aligned} \tag{14}$$

we obtain

$$\frac{\partial u}{\partial \bar{t}} + (u - U) \frac{\partial u}{\partial \bar{x}} = \nu \frac{\partial^2 u}{\partial \bar{x}^2}, \tag{15}$$

so that $w = u - U$ satisfies the same equation in $(\bar{x}, \bar{t})$ as u in (x, t):

$$\frac{\partial w}{\partial \bar{t}} + w \frac{\partial w}{\partial \bar{x}} = \nu \frac{\partial^2 w}{\partial \bar{x}^2}.$$

In this sense (1) is invariant under a Galilean transformation. Bateman used the steady-state solution (13) as $w(\bar{x}, \bar{t})$ in order to show a shock $u(x, t)$ progressing with a velocity U

$$u = U - u_1 \tanh \frac{u_1(x - x_1 - Ut)}{2\nu}.$$

Laws of similarity are also of importance in understanding the joint effects of non-linearity and viscosity in (1). For clarity, consider a solution u of (1) depending on the following parameters:

l = significant length; e.g. size of domain,

u_0 = significant initial amplitude,

ν = viscosity.

Then it is possible to express any solution of (1) in terms of these non-dimensional variables:

$$R = \frac{ul}{\nu}, \qquad R_0 = \frac{u_0 l}{\nu}, \qquad \tau = \frac{t\nu}{l^2}, \qquad \xi = \frac{x}{l}$$

as

$$R = F(R_0, \tau, \xi). \tag{16}$$

This relationship is, of course, general. However it should be compared with the corresponding linear case where it is possible to express the solution as

$$\frac{u}{u_0} = F(\tau, \xi). \tag{17}$$

These results are easily derived by introducing the dimensionless variables τ and ξ in the corresponding equations and seeing what is required of u to make the equations dimensionless. As one example, compare two solutions with the same l but different values of viscosity ν_1, ν_2. In the linear case we can say that the ratio $[u_2/u_{0_2} = u_1/u_{0_1}]_\xi$ when $t_2 = t_1\nu_1/\nu_2$. In the non-linear case however we can only say that u_{0_2} must be adjusted so that $u_{0_2} = u_{0_1}\nu_2/\nu_1$ if the F is to have the same value, and in that case $u_2 = u_1\nu_1/\nu_2$. As might be expacted the non-linear equation cannot give much information under linear transformations.

Various writers (Refs. 5 and 6) have studied the existence and uniqueness of solution

of different types of problems for quasi-linear parabolic equations. For initial value problems it is clear that only one condition is needed at $t = 0$, $u(x, 0) = u_0(x)$. Using this and treating Eq. (1) as an inhomogeneous heat equation, they reduce the problem to an integral equation. Picard iteration procedures can be used to prove existence and uniqueness of solution in the neighborhood of the initial line. The boundary conditions always involve constant values of u(or u_x) on lines x = const. The problem of radiation where conditions like $u(x, t) = f(t)$ have to be considered, has not been treated. It is much harder to prove existence and uniqueness, or to discover necessary and sufficient conditions for this, in the radiation case. Some further remarks about uniqueness will be made in Sec. 5.

5. General Solution of Initial Value Problem. The general solution developed here applies directly to the case when the initial values are known in some domain

$$u(x, 0) = u_0(x), \tag{18}$$

and the boundary conditions are of a simple type. For example, we may have

$$u(x_1, t) = 0, \qquad u(x_2, t) = 0. \tag{19}$$

The solution is supposed to be bounded function having the necessary derivatives.

The general result is: If $\theta(x, t)$ is any solution to the heat equation

$$\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2}, \tag{20}$$

then

$$u(x, t) = -2\nu \frac{\theta_x}{\theta} \tag{21}$$

is a solution to

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

The result can be checked directly by differentiation. However it was derived as follows. Let

$$u = \frac{\partial \phi}{\partial x}, \qquad \phi = \phi(x, t) \tag{22}$$

and substitute in (1). Integrating with respect to x we obtain

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 = \nu \frac{\partial^2 \phi}{\partial x^2}, \tag{23}$$

where the function of integration is omitted. Like the heat equation (23) is invariant under the transformation $x \to ax$, $t \to a^2 t$ (a = constant). This suggests finding solutions of the form

$$\phi(x, t) = F\{\theta(x, t)\} \tag{24}$$

where θ satisfies (20). Introducing (24) into (23) we have

$$F' \cdot \theta_t + \frac{1}{2} F'^2 \cdot \theta_x^2 = \nu\{F''\theta_x^2 + F'\theta_{xx}\}. \tag{25}$$

Hence we obtain the ordinary differential equation for $F(\theta)$

$$\frac{1}{2}\left(\frac{dF}{d\theta}\right)^2 = \nu \frac{d^2F}{d\theta^2}, \tag{26}$$

which has the solution

$$F(\theta) = -2\nu \log (\theta - c_1) + c_2 . \tag{27}$$

Therefore, F is the log of a solution to the heat equation and $u(x, t)$ can be expressed as in (21).

Integrating (21) with respect to x we obtain the equivalent relation

$$\theta(x, t) = C(t) \exp\left(-\frac{1}{2\nu}\int_b^x u(\xi, t)\, d\xi\right), \tag{28}$$

where b is an arbitrary constant and $C = \theta(b, t)$. Without loss of generality b may be normalized to the value zero. The initial values are simply related. If

$$u(x, 0) = u_0(x), \tag{29}$$

then

$$\theta(x, 0) = \theta_0(x) = C_0 \exp\left(-\frac{1}{2\nu}\int_0^x u_0(\xi)\, d\xi\right). \tag{30}$$

Various representations of the solution to the heat equation may be used. A representation suitable for an infinite domain $(-\infty < x < \infty)$ is given by

$$\theta(x, t) = \frac{1}{2(\pi\nu t)^{1/2}}\int_{-\infty}^{+\infty} \exp\left[-\frac{(x-\xi)^2}{4\nu t}\right]\theta_0(\xi)\, d\xi. \tag{31}$$

Thus given θ_0 from (30), θ is found from (31) and $u(x, t)$ from (21). If an integration by parts is carried out in the expression for θ_x (21) becomes

$$u(x, t) = \frac{\int_{-\infty}^{+\infty} \exp\left[-(x-\xi)^2/4\nu t\right] \exp\left[-(2\nu)^{-1}\int_0^\xi u_0(\eta)\, d\eta\right] u_0(\xi)\, d\xi}{\int_{-\infty}^{+\infty} \exp\left[-(x-\xi)^2/4\nu t\right] \exp\left[-(2\nu)^{-1}\int_0^\xi u_0(\eta)\, d\eta\right] d\xi}, \tag{32}$$

an expression for $u(x, t)$ in terms of its initial values.

The uniqueness of the solution to (1) in the domain $(-\infty < x < \infty)$ under the initial conditions (29) can be discussed as follows. Any solution $u(x, t)$ of (1) defines a function $\theta(x, t)$ according to (28), where $C(t)$ can be found such that $\theta(x, t)$ satisfies the heat equation (20). For each u, $C(t)$ is uniquely determined within a multiplicative constant. The initial values of $\theta(x, t)$ depend only on the initial values of $u(x, t)$ according to (30). Now assume there are two solutions u, v of (1) having the same initial values. If $u_0(x)$ is suitably restricted $\theta(x, t)$ is uniquely determined by its initial values. But u, v as solutions of (1) are computed from their corresponding θ according to (21). In this formula the factor $C(t)$ cancels out. Since θ is uniquely determined, u and v are identical for all (x, t) and the solution to (1) is unique.

6. Examples of Solutions. The first example is that of a shock wave approaching a steady state. We choose a step-function as the initial condition where the strength of jump is chosen so that the center of the jump will remain fixed. Then we see how viscosity smoothes even a sharp discontinuity.

The initial conditions are

$$u_0(x) = u_1 \; x < 0, \qquad = -u_1 \; x > 0. \tag{33}$$

From (28), putting $b = 0$

$$\theta_0(x) = C_0 \exp(u_1 x/2\nu) \; x > 0, \qquad = C_0 \exp(-u_1 x/2\nu) \; x < 0. \tag{34}$$

Upon substituting (34) in (31) we obtain

$$\theta(x, t) = \frac{C_0}{2} \exp\left(\frac{u_1^2 t}{4\nu}\right)\left\{2ch\, \frac{u_1 x}{2\nu} + \exp\left(\frac{u_1 x}{2\nu}\right) \operatorname{erf} \frac{x + u_1 t}{2(\nu t)^{1/2}} - \exp\left(-\frac{u_1 x}{2\nu}\right) \operatorname{erf} \frac{x - u_1 t}{2(\nu t)^{1/2}}\right\} \tag{35}$$

Hence from (21) the solution is

$$u(x, t) = -u_1 \frac{2sh(u_1 x/2\nu) + \{\exp[u_1 x/2\nu]\operatorname{erf}[(x + u_1 t)/2(\nu t)^{1/2}] + \exp[-u_1 x/2\nu]\operatorname{erf}[(x - u_1 t)/2(\nu t)^{1/2}]\}}{2ch(u_1 x/2\nu) + \{\exp[u_1 x/2\nu]\operatorname{erf}[(x + u_1 t)/2(\nu t)^{1/2}] - \exp[-u_1 x/2\nu]\operatorname{erf}[(x - u_1 t)/2(\nu t)^{1/2}]\}} \tag{36}$$

It is easily verified that as $t \to 0$ the initial conditions are satisfied if we use

$$\operatorname{erf}(\infty) = 1, \qquad \operatorname{erf}(-\infty) = -1. \tag{37}$$

For large values of t/ν (and $|x| \neq ut$) we can substitute in (36) the asymptotic formulas for erf,

$$\operatorname{erf} z \cong 1 - \frac{1}{\pi^{1/2} z} \exp(-z^2) + \exp(-z^2) 0\left(\frac{1}{z^3}\right), \qquad z > 0. \tag{38}$$

This shows that the approach to the steady state given by Eq. (13) is very rapid. The deviations from the steady state die out like $\exp(-u_1^2 t/4\nu)$.

The passage to the limit $\nu \to 0$ in the solution of (1) is important for determining the behavior of discontinuities in the solution to the equation (3) with $\nu = 0$ from the start. Putting $\nu = 0$ in (36) we have, if $u_1 > 0$

$$u(x, t) = -(\operatorname{sign} x) u_1 ,$$

so that the initial conditions are preserved and we have a stationary shock wave. The invariant quantity across this shock wave is the kinetic energy u_1^2. This is to be taken as the rule for treating discontinuities in the solution to (3), from the viewpoint of an observer at rest relative to the discontinuity. In addition the flow velocity must decrease in passing through the shock. In the case the initial velocities are not chosen as in (33) the situation is the same in a moving coordinate system. The solution for a shock

tends to a quasi-steady state progressing with a certain velocity U, as indicated in Sec. 4. For example if

$$\begin{aligned} u(x, 0) = u_0(x) &= u_3, \qquad x < 0 \\ &= u_4, \qquad x > 0 \qquad \text{where } u_3 > u_4, \end{aligned} \tag{39}$$

we may define a moving coordinate system so that the solution following (33) applies. Let

$$u_1 = (u_3 - U) = -(u_4 - U) \tag{40}$$

or

$$U = \frac{1}{2}(u_3 + u_4). \tag{41}$$

Then according to (15), the solution (36) applies to (39) if x is replaced by $x - Ut$, and u by $u - U$. The speed of propagation of the shock U is the average of the velocities on both sides.

The second example shows the decay of an arbitrary periodic initial disturbance. This corresponds in the turbulence model theory to the decay of free turbulence in a box. The initial and boundary conditions are:

$$u(x, 0) = u_0(x), \qquad 0 \le x \le l \tag{41a}$$

$$u(0, t) = u(l, t) = 0, \qquad t > 0. \tag{41b}$$

From (21) and (30) these conditions induce the following conditions on $\theta(x, t)$

$$\theta(x, 0) = \theta_0(x) = C_0 \exp\left[-\frac{1}{2\nu}\int_0^x u_0(\xi)\, d\xi\right], \tag{43a}$$

$$\theta_x(0, t) = \theta_x(l, t) = 0 \tag{43b}$$

The problem for the heat equation specified by conditions (43) has a unique (bounded) solution. We can represent the solution to the heat equation in a standard way by a Fourier series in x whose coefficients are exponentials in t

$$\theta(x, t) = A_0 + \sum_{n=1}^{\infty} \exp\left[-\nu \frac{n^2\pi^2}{l^2} t\right] A_n \cos \frac{n\pi x}{l}, \tag{44}$$

so that (43b) is satisfied. The coefficients A_0, A_n are determined at $t = 0$ as

$$A_0 = \frac{1}{l}\int_0^l \theta_0(x)\, dx = \frac{C_0}{l}\int_0^l \exp\left[-\frac{1}{2\nu}\int_0^x u_0(\xi)\, d\xi\right] dx \tag{45a}$$

$$A_n = \frac{2}{l}\int_0^l \theta_0(x) \cos\frac{n\pi x}{l}\, dx = \frac{2C_0}{l}\int_0^l \exp\left[-\frac{1}{2\nu}\int_0^x u_0(\xi)\, d\xi\right] dx \tag{45b}$$

Hence from (21) the solution is

$$u(x, t) = \frac{2\nu\pi}{l} \frac{\sum_{n=1}^{\infty} \exp\left[-\nu n^2\pi^2 t/l^2\right] n A_n \sin(n\pi x/l)}{A_0 + \sum_{n=1}^{\infty} \exp\left[-\nu n^2\pi^2 t/l^2\right] A_n \cos(n\pi x/l)} \tag{46}$$

For large values of time t only the first term in the numerator remains so that

$$u(x, t) \cong \frac{2\nu\pi}{l} \frac{A_1}{A_0} \exp\left[-\nu \frac{\pi^2}{l^2} t\right] \sin \frac{\pi x}{l} \tag{47}$$

This may be contrasted with the solution to corresponding linear problems at large times

$$\theta(x, t) = B_1 \exp\left[-\nu \frac{\pi^2}{l^2} t\right] \sin \frac{\pi x}{l}, \tag{48}$$

where

$$B_1 = \frac{2}{l} \int_0^l u_0(x) \sin \frac{\pi x}{l}\, dx.$$

The solutions (47) and (48), are seen to have the same form in dependence on (x, t), but different amplitudes. The similarity of the solutions is an expression of the fact that when the amplitudes are small the non-linear equation behaves like the linear one. However the decay process over intermediate ranges of time is considerably different. We can find out something about this process by considering special cases. For example, consider a simple sine wave

$$u_0(x) = u_0 \sin \frac{\pi x}{l} \cdot \tag{49}$$

The coefficients are explicitly evaluated in this case as

$$A_0 = \frac{C_0}{l} \int_0^l \exp\left[-\frac{u_0 l}{2\pi\nu}\left(1 - \cos\frac{\pi x}{l}\right)\right] dx = C_0 \exp\left[-\frac{u_0 l}{2\pi\nu}\right] I_0\left(\frac{u_0 l}{2\pi\nu}\right), \tag{50}$$

$$A_n = \frac{2C_0}{l} \int_0^l \exp\left[-\frac{u_0 l}{2\pi\nu}\left(1 - \cos\frac{\pi x}{l}\right)\right] \cos\frac{n\pi x}{l}\, dx = 2C_0 \exp\left[-\frac{u_0 l}{2\pi\nu}\right] I_n\left(\frac{u_0 l}{2\pi\nu}\right), \tag{51}$$

so that (46) becomes

$$u(x, t) = \frac{4\nu\pi}{l} \frac{\sum_{n=1}^{\infty} \exp(-\nu n^2\pi^2 t/l^2) n I_n(u_0 l/2\pi\nu) \sin(n\pi x/l)}{I_0(u_0 l/2\pi\nu) + 2 \sum_{n=1}^{\infty} \exp(-\nu^2 n^2 \pi^2 t/l^2) I_n(n_0 l/2\pi\nu) \cos(n\pi x/l)} \cdot \tag{52}$$

The conditions at $t = 0$ are satisfied by (52) for

$$I_0(z) + 2 \sum_{n=1}^{\infty} I_n(z) \cos nx = \exp(z \cos x). \tag{53}$$

The significant parameter occurring in the solution is the Reynolds number R_0 based on the size of the box

$$R_0 = \frac{u_0 l}{\nu} \cdot \tag{54}$$

The complexity of (52) is due to excitation of higher frequencies because of the non-linearity of (1). The same excitation of higher frequencies was noted by Burgers by directly splitting the solution of (1) into the form

$$u(x, t) = \sum_{n=1}^{\infty} E_n(t) \sin nx$$

and observing the coupling between the E_n. This can be contrasted with solutions to the heat equation under the same initial conditions

$$\theta(x, t) = u_0 \exp\left[-\nu \frac{\pi^2}{l^2} t\right] \sin \frac{\pi x}{l} \tag{55}$$

which, for all time, shows only the fundamental frequency. This is typical of linear equations as is the linear dependence on the amplitude of the initial disturbance. The solution (52) emphasizes the non-linear dependence on the initial conditions. The initial amplitude enters through R_0. As $R_0 \to 0$, $u(x, t) = \theta(x, t) + 0(R_0)$. This estimate shows to what approximation the non-linear terms can be neglected and emphasizes that the dimensionless parameter R_0 (not merely u_0) should be small ($R_0 \ll 1$). For large values of R_0 all the $I_n(R_0/2\pi)$ are almost equal so that large changes in R_0 (increases in u_0) produce relatively little effect on the solution. Asymptotically we have

$$I_n\left(\frac{R_0}{2\pi}\right) \cong \frac{\exp(R_0/2\pi)}{R_0^{1/2}}\left\{1 - \frac{4n^2 - 1}{8R_0/2\pi} + O\left(\frac{1}{R_0^2}\right)\right\} \tag{56}$$

As the first approximation for large R_0 the $I_n(R_0/2\pi)$ may be cancelled in (52) to give

$$u(x, t) \doteq \frac{4\nu\pi}{l} \frac{\sum_{n=1}^{\infty} \exp(-\nu n^2\pi^2 t/l^2) n \sin(n\pi x/l)}{1 + 2\sum_{n=1}^{\infty} \exp(-\nu n^2\pi^2 t/l^2) \cos(n\pi x/l)}\,. \tag{57}$$

Equation (57) is an approximation which should describe what happens for t greater than some $t_1 > 0$. Equation (57) can be put in a simpler form by using an identity for theta functions (Ref. 7, p. 489)

$$u(x, t) = -\frac{2\nu\pi}{l} \sum_{n=1}^{\infty} \frac{(-)^n \sin(n\pi x/l)}{\sinh \nu(n\pi^2 t/l^2)} \tag{58}$$

For large R_0, Eq. (58) gives the spectrum, involving all higher frequencies, explicitly. The solution is independent of the initial amplitude and the spectrum damps exponentially with the first power of the wave number for large n. For small n and $\nu n^2 t/l^2 \ll 1$ the coefficients depend on $1/n$. The dissipation is proportional to u_x^2 and is thus independent of n for the first few n. The exponential cut off for large n assures a finite total dissipation. For $0 \leq x < l$ (58) can be written in another form,

$$u(x, t) \doteq \frac{x}{t} + 2\frac{l}{t} \sum_{m=1}^{\infty} (-)^m \frac{sh(mx/\nu t)}{sh(ml/\nu t)}\,. \tag{59}$$

For $x/\nu t$ large, the series can be approximately summed as

$$u(x, t) \doteq \frac{l}{t}\left\{\tanh\left(\frac{l - x}{2\nu t}\right) - \left(1 - \frac{x}{l}\right)\right\}, \tag{60}$$

a form which shows a steep front near $x = l$. The general picture presented by the above considerations is the following. The initial sine wave (49) shows after the first instant a tendency to develop a steep front near $x = l$, if R_0 is sufficiently large. After a while this steep front broadens and dies out until at the end only a sine wave remains. This sine wave has an amplitude which is smaller than that of the corresponding linear problem because of the increased dissipation over the intermediate ranges of t. It is clear that similar considerations apply to any initial distribution of the same general form as a sine wave.

7. Concluding remarks. The simple examples which have been worked out are intended to illustrate some general features of the interaction of non-linearity and viscosity. The main effects are always the same, namely, steepening of the velocity profiles by non-linearity and prevention of discontinuities, diffusion of momentum and dissipation of energy by viscosity. Under different interpretations these effects are considered responsible for the formation of steep but continuous shock wave fronts and for the finite dissipation and feeding of energy through the spectrum. As an example of different interpretations consider a velocity distribution with a steep front. The discontinuous front of non-viscous flow has infinite dissipation when considered in viscous flow and it contributes terms like $1/n$ to the spectrum. When viscosity is considered from the outset the front is steep but continuous, the dissipation is finite and independent of ν; while the front is steep there are some terms like $1/n$ in the spectrum but the spectrum dies out like exp $(-n)$ for large n. Another important general feature is the non-linear dependence of the solution on a characteristic Reynolds number $R_0 = u_0 l/\nu$, $u_0 x/\nu$. For low $R_0(R_0 \ll 1)$ the non-linearity is not important and the solution behaves like the solution to the corresponding heat equation but as R_0 increases the solution changes very much. For large R_0 it is typical that there are ranges of (x, t) for which the solution depends very little on the variations in R_0. Part of the problem for the future is a more precise determination of the ranges in which the various approximations are valid.

The same type of result applies to some special solutions in higher dimensions. For the equation in two or three dimensions which is analogous to (1)

$$\mathbf{q}_t + \mathbf{q} \cdot \nabla \mathbf{q} = \nu \nabla^2 \mathbf{q} \tag{61}$$

it can be verified that the method of Sec. 5 applies. If $\theta(x, t)$ is any solution of

$$\theta_t = \nu \nabla^2 \theta, \tag{62}$$

then

$$\mathbf{q} = -2\nu \nabla (\log \theta) \tag{63}$$

is a solution to (61). It should be noted that (63) gives an irrotational flow field.

For future work it seems worthwhile to investigate simple solutions further and study the radiation problems. Then the three-dimensional cases may also be studied.

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COMPRESSIBLE FLOWS WITH DEGENERATE HODOGRAPHS*

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1. Introduction. The theory of compressible perfect fluids has developed slowly because the basic equations are non-linear. Thus it has been profitable to consider special examples, such as will be studied here. The present problem originates in the study of steady, two-dimensional, isentropic, irrotational flow. If there is a biunique mapping of the physical plane onto the hodograph plane, then the equation for the velocity potential function can be linearized by a Legendre transformation [5].** This draws attention to the case in which the transformation may fail because the mapping is nowhere biunique. This suggests the problem to investigate all three-dimensional flows whose images in the hodograph space, for Cartesian coordinates and velocity components, are curves or surfaces. Such flows are sometimes said to be "lost" [10] by contrast with the nomenclature used here. By analogy with the usage in [5], flows with one- or two-dimensional hodographs will be called *simple* or *double* waves. The hodograph of a flow will be called *degenerate* when it has fewer dimensions than the original physical space.

The problem can also be motivated as follows. Among the most familiar compressible flows are Prandtl-Meyer expansion around a corner or curved wall [8]; Busemann's cylindrical or "swept-back" flow produced by superposition of plane flow and uniform flow normal to that plane [2]; Taylor-Maccoll axisymmetric flow about a cone [1, 3, 7, 11]; and Busemann's general conical flows [2]. In these examples the loci of particles of equal velocity are planes or straight lines, so their hodographs are degenerate. The question arises, whether this enumeration is exhaustive.

In this paper the flow will be assumed to be steady, isentropic, and irrotational. Characterizations of one- and two-dimensional hodographs will be developed, and generalizations will be found for the properties of the examples mentioned above. As an example the construction of flows with axisymmetric degenerate hodographs will be considered.

Some aspects of this problem have been considered by Germain [6]. M. H. Martin has also made an unpublished investigation along these lines. The construction of all axisymmetric flows with degenerate hodographs was studied by Bateman and later by Stewart [10]. Opatowski [9] has discussed very concisely the more general problem to determine those flows for which the covariant velocity components in some curvilinear coordinate system depend only on two coordinates.

2. Fundamental equations. Compressible perfect flow obeys the equations of motion

$$u_i \, \partial u_k/\partial x^i = -\rho^{-1} \, \partial p/\partial x^k \tag{2.1}$$

and the equation of continuity

$$\partial(\rho u_i)/\partial x^i = 0. \tag{2.2}$$

*Received Aug. 15, 1950. Presented to the American Physical Society, Feb. 1, 1947.

**Numbers in brackets designate papers listed at the end of this note.

The x^i ($i = 1, 2, 3$) denote Cartesian coordinates in the physical space, u_i velocity components, ρ density, and p pressure. The convention that every pair of repeated sub- or superscripts implies summation over their range has been adopted. For irrotational flow

$$\partial u_i/\partial x^j = \partial u_j/\partial x^i. \tag{2.3}$$

For isentropic flow

$$p/p_0 = (\rho/\rho_0)^\gamma \tag{2.4}$$

for certain reference values p_0 and ρ_0, and $\gamma = c_p/c_v$, the ratio of the specific heats at constant pressure and volume. Equations (2.1), (2.3), and (2.4) imply Bernoulli's equation

$$\tfrac{1}{2}u_iu_i + a^2/(\gamma - 1) = \tfrac{1}{2}c^2, \tag{2.5}$$

where

$$a^2 = dp/d\rho = \gamma p/\rho \tag{2.6}$$

is the square of the speed of sound, and the constant c is the limiting speed of flow. By (2.3) there exists a velocity potential function φ such that

$$u_i = \partial\varphi/\partial x^i. \tag{2.7}$$

By (2.1, (2.3), (2.4), and (2.6)

$$a^2\, \partial\rho/\partial x^k = -\rho u_i\, \partial u_i/\partial x^k, \tag{2.8}$$

and by (2.2), (2.7), and (2.8)

$$(a^2\delta_{ij} - u_iu_j)\, \partial^2\varphi/\partial x^i\, \partial x^j = 0, \tag{2.9}$$

where Kronecker's delta, $\delta_{ij} = 1$ (0) if $i = (\neq)\ j$, and where by (2.5)

$$a^2 = \tfrac{1}{2}(\gamma - 1)(c^2 - u_iu_i). \tag{2.10}$$

3. Degenerate Legendre transformations. The transformation $x^i \to u_i$ maps a three-dimensional region of the physical space onto an n-dimensional region of the hodograph space if and only if

$$\partial\varphi/\partial x^i = u_i = u_i(\mu), \tag{3.1}$$

the functions $\mu^\alpha(x)$ ($\alpha = 1, \cdots, n$) are independent, and n of the functions $u_i(\mu)$ are also independent. Hence the

$$\text{rank of } \|\, \partial\mu^\alpha/\partial x^i \,\| = \text{rank of } \|\, \partial u_i/\partial\mu^\alpha \,\| = n. \tag{3.2}$$

Disregard $n = 0$ (uniform flow). Then $n = 1$ or 2 for degenerate hodographs. Let

$$k = \varphi - x^iu_i \tag{3.3}$$

By (3.1) and (3.3) $\partial k/\partial x^j = -x^i(\partial u_i/\partial\mu^\alpha)(\partial\mu^\alpha/\partial x^j)$. Accordingly, the Jacobian matrix of k and μ^α has the same rank as that of μ^α alone, so $k = k(\mu)$. (3.3) becomes

$$\varphi = x^iu_i(\mu) + k(\mu), \tag{3.4}$$

and by (3.1) and (3.4) $(x^i\, \partial u_i/\partial\mu^\alpha + \partial k/\partial\mu^\alpha)\, \partial\mu^\alpha/\partial x^j = 0$. By (3.2)

$$x^i\, \partial u_i/\partial\mu^\alpha + \partial k/\partial\mu^\alpha = 0. \tag{3.5}$$

By (3.5)

$$(x^i \, \partial^2 u_i/\partial\mu^\alpha \, \partial\mu^\beta + \partial^2 k/\partial\mu^\alpha \, \partial\mu^\beta) \, \partial\mu^\beta/\partial x^j = -\partial u_j/\partial\mu^\alpha. \tag{3.6}$$

By (3.2) this implies

$$\text{rank of } \| \, x^i \, \partial^2 u_i/\partial\mu^\alpha \, \partial\mu^\beta + \partial^2 k/\partial\mu^\alpha \, \partial\mu^\beta \, \| = n. \tag{3.7}$$

Hereafter assume that $u_i(\mu)$ and $k(\mu)$ have been chosen to satisfy (3.7). Then (3.5) can be inverted to yield $\mu^\alpha(x)$.

By (3.1) a point on the hodograph is determined by setting $\mu^\alpha = \mu_0^\alpha$. The set of points in the physical space which is mapped onto $u_i(\mu_0)$ will be called its *prototype.* (3.5) implies

THEOREM 3.1: *If the coordinate axes of the physical and hodograph spaces are parallel, the prototype in the physical space of a point, P, of a one (two) dimensional degenerate hodograph, H, is contained in a plane (line) parallel to the plane (line) normal to H at P.*

So far ϕ has only been compelled to yield a degenerate map. For a compressible flow (2.9) and (2.10) must also be satisfied. In (2.9) $\partial^2\varphi/\partial x^i \, \partial x^i$ is required. By (3.1)

$$\partial^2\varphi/\partial x^i \, \partial x^i = (\partial u_i/\partial\mu^\alpha)(\partial\mu^\alpha/\partial x^i), \tag{3.8}$$

where $\partial\mu^\alpha/\partial x^i$ must be obtained from (3.6).

4. Simple waves. When $n = 1$, (3.6) to (3.8) yield

$$\partial^2\varphi/\partial x^i \partial x^i = -u_i' u_i'/(x^m u_m'' + k''), \tag{4.1}$$

where primes denote ordinary derivatives with respect to μ^1. By (2.9), (2.10), and (4.1)

$$a^2 u_i' u_i' = (u_i u_i')^2. \tag{4.2}$$

If s is arc-length measured from some point of the hodograph curve and q is speed of flow

$$(s')^2 = u_i' u_i', \tag{4.3}$$

$$q^2 = u_i u_i. \tag{4.4}$$

Now (4.2) implies

$$a^2 = \tfrac{1}{2}(\gamma - 1)(c^2 - q^2) = q^2(dq/ds)^2. \tag{4.5}$$

Construct a cone, K, with vertex, V, at the origin of the hodograph space and passing through the hodograph curve C. When K is developed onto a plane, C will be deformed into a plane curve C' to which (4.5) also applies. Hence C' is the familiar epicycloid of the Prandtl-Meyer flow around a corner. Accordingly, C will be called a *conically deformed Prandtl-Meyer epicycloid.*

THEOREM 4.1: *The hodograph of a simple wave consists of arcs of conically deformed Prandtl-Meyer epicycloids. Conversely, a sufficiently small arc of a conically deformed Prandtl-Meyer epicycloid, on which the direction of the tangent vector varies continuously, is the hodograph of a simple wave.*

For the converse, construct a velocity field with the prescribed hodograph. Suppose that for $A \leqq \mu^1 \leqq B$, $u_i = u_i(\mu^1)$ is an arc of a conically deformed Prandtl-Meyer

epicycloid, the u_i being of class C^1 on AB. Let $f(\mu^1)$ be an arbitrary function continuous on AB, and let

$$x^i(\mu^1) = x_A^i + \int_A^{\mu^1} u_i(\mu) f(\mu)\, d\mu,$$

where the x_A^i are constants. To prevent the curve $x^i = x^i(\mu^1)$ from intersecting itself, decrease the interval AB, if necessary. As suggested by Theorem 3.1, through each point $x^i(\mu^1)$ construct a plane normal to $u_i'(\mu^1)$, and assign $u_i(\mu^1)$ to every point of this plane. By making the interval AB small enough, and by considering only a region close enough to the curve $x^i = x^i(\mu^1)$, a continuous single-valued velocity vector field can be obtained. Finally, by constructing in this vector field a family of streamlines close to the streamline $x^i = x^i(\mu^1)$, a stream tube, and hence a flow with the desired hodograph will be produced.

If (4.5) is interpreted as an equation of a plane curve, it is clear that *in a simple wave the flow must be supersonic*. Discontinuities in the second or higher order derivatives of $u_i(\mu^1)$ are propagated along prototype planes. Thus the Mach cone at any point, P, of a simple wave must be tangent to the prototype plane, Π, through P, and the streamline, S, through P intersects Π at the Mach angle.

The reader may verify the following assertions. (1) Sufficiently small arcs of any curve with continuous curvature can be arcs of streamlines of simple waves. (2) For a sufficiently small range of values of μ^1 any one parameter family of planes $A_m(\mu^1)x^m + B(\mu^1) = 0$ can be chosen to be the prototype planes of a simple wave, provided $A_m(\mu^1)$ and $B(\mu^1)$ are of class C^1, and provided that not all of these planes are parallel.

As an example for this section, consider a simple wave, W, in which the envelope of the prototype planes is a cylinder, S. By Theorem 3.1 the hodograph, H, of W is a plane curve. Orient axes so H lies in u_3 = constant, and let $Q^2 = q^2 - u_3^2$. (4.5) becomes $\frac{1}{2}(\gamma - 1)[(c^2 - u_3^2) - Q^2] = (QdQ/ds)^2$. This defines an epicycloid obtained by shrinking the generating circles of the usual Prandtl-Meyer epicycloid by a factor $(1 - u_3^2/c^2)^{1/2}$. As indicated in Sec. 5, W is a swept-back version of Prandtl-Meyer flow.

5. Double waves. When $n = 2$ let $\Delta = \det \| x^m\, \partial^2 u_m/\partial\mu^\alpha\, \partial\mu^\beta + \partial^2 k/\partial\mu^\alpha\, \partial\mu^\beta \|$, where $\Delta \neq 0$ by (3.7). By (3.6)

$$\Delta \partial\mu^\beta/\partial x^k = (-1)^{\alpha+\beta+1}(x^m\, \partial^2 u_m/\partial\mu^{\alpha+1}\, \partial\mu^{\beta+1} + \partial^2 k/\partial\mu^{\alpha+1}\, \partial\mu^{\beta+1})\, \partial u_k/\partial\mu^\alpha, \tag{5.1}$$

where α is summed, but not β, and where $\alpha + 1$ and $\beta + 1$ are reduced mod 2. For fixed μ^α the solutions x^i of (3.5) lie on a line. Let $\nu^i(\mu)$ be parallel to this line, so

$$\nu^i\, \partial u_i/\partial\mu^\alpha = 0, \tag{5.2}$$

and let $x_0^i(\mu)$ be a particular solution of (3.5). The general solution is

$$x^i = x_0^i(\mu) + r\nu^i(\mu), \tag{5.3}$$

where the parameter r is independent of μ^α. Now (5.1) becomes

$$\Delta\partial\mu^\beta/\partial x^k = (-1)^{\alpha+\beta+1}[(r\nu^m + x_0^m)\partial^2 u_m/\partial\mu^{\alpha+1}\partial\mu^{\beta+1} + \partial^2 k/\partial\mu^{\alpha+1}\partial\mu^{\beta+1}]\partial u_k/\partial\mu^\alpha \tag{5.4}$$

for β not summed. By (2.9) and (3.7)

$$\begin{aligned}(a^2\delta^{ij} - u_i u_j)(\partial u_i/\partial\mu^\alpha)(\partial u_j/\partial\mu^\beta)(-1)^{\alpha+\beta}\\ \cdot[(r\nu^m + x_0^m)\partial^2 u_m/\partial\mu^{\alpha+1}\partial\mu^{\beta+1} + \partial^2 k/\partial\mu^{\alpha+1}\partial\mu^{\beta+1}] = 0\end{aligned} \tag{5.5}$$

with both α and β summed. Since r is independent of μ^α

$$(a^2\delta^{ij} - u_iu_j)(\partial u_i/\partial\mu^\alpha)(\partial u_j/\partial\mu^\beta)(-1)^{\alpha+\beta}\nu^m\partial^2 u_m/\partial\mu^{\alpha+1}\partial\mu^{\beta+1} = 0, \tag{5.6}$$

$$\begin{aligned}(a^2\delta^{ij} - u_iu_j)(\partial u_i/\partial\mu^\alpha)(\partial u_j/\partial\mu^\beta)(-1)^{\alpha+\beta}(x_0^m\partial^2 u_m/\partial\mu^{\alpha+1}\partial\mu^{\beta+1}& \\ + \partial^2 k/\partial\mu^{\alpha+1}\partial\mu^{\beta+1}) = 0.&\end{aligned} \tag{5.7}$$

Now let $g_{\alpha\beta}$ be the covariant metric tensor of the hodograph surface and $b_{\alpha\beta}$ its second fundamental tensor. By definition

$$g_{\alpha\beta} = (\partial u_i/\partial\mu^\alpha)(\partial u_i/\partial\mu^\beta), \tag{5.8}$$

$$b_{\alpha\beta} = \nu^m(\partial^2 u_m/\partial\mu^\alpha\partial\mu^\beta), \tag{5.9}$$

where ν^m is a unit normal to the surface, i.e.

$$\nu^i\nu^i = 1. \tag{5.10}$$

Also

$$u_k\,\partial u_k/\partial\mu^\alpha = q\,\partial q/\partial\mu^\alpha = aM\,\partial q/\partial\mu^\alpha, \tag{5.11}$$

where $M = q/a$. Write $\partial q/\partial\mu^\alpha = q_{,\alpha}$, where the subscript $_{,\alpha}$ denotes the covariant derivative with respect to μ^α and based on $g_{\alpha\beta}$. Then (5.6) becomes

$$(g_{\alpha\beta} - M^2 q_{,\alpha}q_{,\beta})(-1)^{\alpha+\beta}b_{\alpha+1\,\beta+1} = 0. \tag{5.12}$$

A particular solution of (3.5) is

$$x_0^i = -(\partial k/\partial\mu^\gamma)g^{\gamma\delta}(\partial u_i/\partial\mu^\delta), \tag{5.13}$$

where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$. Since the Christoffel symbols of the first kind, based on $g_{\alpha\beta}$, are $[\alpha\beta, \gamma] = (\partial u_i/\partial\mu^\gamma)(\partial^2 u_i/\partial\mu^\alpha\,\partial\mu^\beta)$, the second covariant derivative of k becomes

$$k_{,\alpha\beta} = \partial^2 k/\partial\mu^\alpha\,\partial\mu^\beta - (\partial k/\partial\mu^\gamma)g^{\gamma\delta}(\partial u_i/\partial\mu^\delta)(\partial^2 u_i/\partial\mu^\alpha\,\partial\mu^\beta). \tag{5.14}$$

Hence (5.7), (5.8), (5.11), and (5.14) imply

$$(g_{\alpha\beta} - M^2 q_{,\alpha}q_{,\beta})(-1)^{\alpha+\beta}k_{,\alpha+1\beta+1} = 0. \tag{5.15}$$

(5.12) is a second order quasilinear partial differential equation for three functions. To determine $u_i(\mu)$ requires two more equations, which may be obtained by assigning a special form to the coefficient tensor $g_{\alpha\beta} - M^2q_{,\alpha}q_{,\beta}$. The resulting systems are classified according to the nature of the characteristic curves of their integral surfaces.

A characteristic is a curve on which the coordinate functions, their first partial derivatives, and hence the metric tensor are continuous, while the components of the second fundamental tensor may have discontinuities. Suppose $\partial u_i/\partial\mu^\alpha$ are known along $\mu^\alpha = \mu^\alpha(t)$ on $u_i = u_i(\mu)$. By (5.9) the *strip conditions* $d(\partial u_m/\partial\mu^\alpha)/dt = (\partial^2 u_m/\partial\mu^\alpha\,\partial\mu^\beta)\,d\mu^\beta/dt$ imply

$$b_{\alpha\beta}\,d\mu^\beta/dt = \nu^m\,d(\partial u_m/\partial\mu^\alpha)/dt. \tag{5.16}$$

Then $b_{\alpha\beta}$ fails to be uniquely determined along $\mu^\alpha = \mu^\alpha(t)$ by (5.12) and (5.16) only if

$$(g_{\alpha\beta} - M^2q_{,\alpha}q_{,\beta})(d\mu^\alpha/dt)(d\mu^\beta/dt) = 0. \tag{5.17}$$

This defines the characteristic directions $d\mu^\alpha/dt$. By (5.14), if $u_i(\mu)$ are known, then (5.15) is a linear partial differential equation for k which also has the characteristic directions (5.17). Equations (5.12) and (5.15) will be said to be of *hyperbolic*, *parabolic*, or *elliptic* type wherever $\Omega = \det \| g_{\alpha\beta} - M^2 q_{,\alpha} q_{,\beta} \| < 0, = 0, > 0$. Hereafter (5.12) and (5.15) will be assumed to be hyperbolic. Cohn [4] has constructed a double wave of hyperbolic type and has also given simple canonical forms for the hodograph equations for both the hyperbolic and elliptic cases.

If s_c is the arc-length of a characteristic, then (5.17) becomes

$$q^2(dq/ds_c)^2 = a^2, \tag{5.18}$$

which is identical with (4.5). Hence

THEOREM 5.1: *The characteristics of the hodographs of double waves are composed of arcs of conically deformed Prandtl-Meyer epicycloids, i.e. of one-dimensional hodographs.*

By (5.18) the component of $q_{,\alpha}$ along either characteristic is $q_{,\alpha}\, d\mu^\alpha/ds_c = \pm a/q = \pm 1/M$. Hence

THEOREM 5.2: *On the hodographs of double waves the curves of constant speed and their orthogonal trajectories bisect the angles between the characteristics.*

Curves $x^i = x^i(t)$ (other than prototype lines) in the physical space are mapped onto curves $\mu^\alpha = \mu^\alpha(t)$ on the hodograph. It is convenient to know the relation between tangent vectors of a pair of corresponding curves. By (3.5), (5.3), (5.13), and (5.14)

$$(dx^i/dt)(\partial u_i/\partial\mu^\alpha) + [r(t)b_{\alpha\beta} + k_{,\alpha\beta}](d\mu^\beta/dt) = 0 \tag{5.19}$$

for some $r(t)$. Unless $\det \| rb_{\alpha\beta} + k_{,\alpha\beta} \| = 0$, this determines $d\mu^\beta/dt$. Conversely, if the curve $\mu^\alpha = \mu^\alpha(t)$ is given on a hodograph surface, its prototype is the ruled surface

$$x^i(r, t) = x_0^i(t) + [r - A(t)]\nu^i(t), \tag{5.20}$$

where ν^i is a unit normal to the hodograph, $x_0^i(t)$ is defined by (5.13), and $A(t)$ by $dA/dt = \nu^i\, dx_0^i/dt$ (to make the curves r = constant orthogonal to the rulings). For (5.20) an analog of (5.19) is

$$(\partial x^i/\partial t)(\partial u_i/\partial\mu^\alpha) + [(r - A)b_{\alpha\beta} + k_{,\alpha\beta}](d\mu^\beta/dt) = 0. \tag{5.21}$$

Since $\nu^i\, \partial x^i/\partial t = 0$, (5.21) implies

$$\partial x^i/\partial t = -(\partial u_i/\partial\mu^\alpha)g^{\alpha\beta}[(r - A)b_{\alpha\beta} + k_{,\alpha\beta}]\, d\mu^\beta/dt. \tag{5.22}$$

In general, the direction of $\partial x^i/\partial t$ will vary with r along a ruling, so (5.20) need not be developable. This raises the question, what curves on the hodograph have developable prototypes? Since $\partial^2 x^i/\partial r^2 = 0$, (5.20) will be developable if and only if

$$\det \| \partial x^i/\partial r, \quad \partial x^i/\partial t, \quad \partial^2 x^i/\partial r\partial t \| = 0. \tag{5.23}$$

Since ν^i and $\partial u_i/\partial\mu^\alpha$ are linearly independent, by (5.20) and (5.22) (5.23) is equivalent to

$$(Cb_{\alpha\beta} + Dk_{,\alpha\beta})\, d\mu^\beta/dt = 0 \tag{5.24}$$

for some $C(t)$ and $D(t)$ not both zero. By (5.12) and (5.15)

$$(g_{\alpha\beta} - M^2 q_{,\alpha} q_{,\beta})(-1)^{\alpha+\beta}(Cb_{\alpha+1\beta+1} + Dk_{,\alpha+1\beta+1}) = 0. \tag{5.25}$$

Hereafter assume $Cb_{\alpha\beta} + Dk_{,\alpha\beta} \neq 0$ for some α and β. Then (5.24) and (5.25) imply (5.17), i.e. $\mu^\alpha = \mu^\alpha(t)$ is a characteristic.

Next, show that the prototypes of both families of characteristics are developable. Let $\mu^\alpha = \mu^\alpha_\epsilon(t)$ define one characteristic from each family through a given point P of the hodograph. At P, by (5.17)

$$2(g_{\alpha\beta} - M^2 q_{,\alpha} q_{,\beta}) = (-1)^{\alpha+\beta}[(d\mu_1^{\alpha+1}/dt)(d\mu_2^{\beta+1}/dt) + (d\mu_1^{\beta+1}/dt)(d\mu_2^{\alpha+1}/dt)]f$$

for some $f \neq 0$. Then by (5.12) and (5.15)

$$b_{\alpha\beta}(d\mu^\alpha_{\epsilon+1}/dt)(d\mu^\beta_\epsilon/dt) = k_{,\alpha\beta}(d\mu^\alpha_{\epsilon+1}/dt)(d\mu^\beta_\epsilon/dt) = 0, \tag{5.26}$$

where ϵ is not summed. Since these have non-trivial solutions $d\mu^\alpha_{\epsilon+1}/dt$, (5.26) implies (5.24) for $\mu^\alpha = \mu^\alpha_\epsilon$ and some $C = C_\epsilon$ and $D = D_\epsilon$ not both zero.

Now investigate the relation between tangents to characteristics and unit normals n^i_ϵ to prototypes $x^i = x^i_\epsilon(r, t)$ of characteristics $\mu^\alpha = \mu^\alpha_\epsilon(t)$. By (5.20) $n^i_\epsilon v^i = n^i_\epsilon\, \partial x^i_\epsilon/\partial r = 0$. For some A^α_ϵ

$$n^i_\epsilon = A^\alpha_\epsilon\, \partial u_i/\partial\mu^\alpha. \tag{5.27}$$

Since $n^i_\epsilon\, \partial x^i_\epsilon/\partial t = 0$, then by (5.25) $A^\alpha_\epsilon[(r - A)b_{\alpha\beta} + k_{,\alpha\beta}]\, d\mu^\beta_\epsilon/dt = 0$. Since $x^i = x^i_\epsilon(r, t)$ is developable, it must be possible to choose A^α_ϵ so that n^i_ϵ does not vary on a ruling. Thus A^α_ϵ may be assumed independent of r, so

$$A^\alpha_\epsilon b_{\alpha\beta} d\mu^\beta_\epsilon/dt = A^\alpha_\epsilon k_{,\alpha\beta} d\mu^\beta_\epsilon/dt = 0. \tag{5.28}$$

If for some $c(t)$ and $d(t)$

$$\det \| cb_{\alpha\beta} + dk_{,\alpha\beta} \| \neq 0, \tag{5.29}$$

then by (5.26) and (5.28) $A^\alpha_\epsilon = g d\mu^\alpha_{\epsilon+1}/dt$ for some g. By (5.27)

$$n^i_\epsilon = g(\partial u_i/\partial\mu^\alpha)(d\mu^\alpha_{\epsilon+1}/dt), \tag{5.30}$$

i.e. the tangent to $\mu^\alpha = \mu^\alpha_{\epsilon+1}$ is normal to the prototype of $\mu^\alpha = \mu^\alpha_\epsilon$. These considerations and elementary calculation yield

THEOREM 5.3: *If for a double wave* $\det \| cb_{\alpha\beta} + dk_{,\alpha\beta} \| \neq 0$ *for some c and d, and if* $b_{\alpha\beta}$ *and* $k_{,\alpha\beta}$ *are linearly independent*:

(1) *The characteristics are the only curves on the hodograph with developable prototypes.*

(2) *The tangent at any point of a characteristic is normal, at points of the corresponding ruling, to the prototype of the other characteristic through those points.*

(3) *The Mach cone at any point of the prototype of a characteristic is tangent to the prototype.*

(4) *The streamlines intersect the prototypes of characteristics at the Mach angle.*

For the omitted cases, first suppose $\det \| b_{\alpha\beta} \| \equiv 0$, which includes the case $\det \| cb_{\alpha\beta} + dk_{,\alpha\beta} \| \equiv 0$. Then the hodograph is developable, so one family of lines of curvature consists of rulings. There exist b_α such that $b_{\alpha\beta} = b_\alpha b_\beta$. Suppose b_α is non-null. Let n^α be a non-trivial solution of $b_\alpha n^\alpha = 0$. Then $b_{\alpha\beta} n^\beta = 0$, so n^α is tangent to a line of curvature of curvature zero, i.e. a ruling. Since some $b_{\alpha\beta} \neq 0$, the lines of curvature of the hodograph are uniquely determined. On the other hand, by (5.12) $(g_{\alpha\beta} - M^2 q_{,\alpha} q_{,\beta})(-1)^{\alpha+\beta} b_{\alpha+1} b_{\beta+1} = 0$, so $(g_{\alpha\beta} - M^2 q_{,\alpha} q_{,\beta}) n^\alpha n^\beta = 0$, and n^α is a characteristic vector. Hence one family of characteristics must consist of rulings. By Theorem

5.1 a plane characteristic is a Prandtl-Meyer epicycloid, not a straight line. Hence $b_\alpha = 0$, so $b_{\alpha\beta} = 0$, and the hodograph must be in a plane, which may be assumed to be $u_3 =$ constant. Thus prototypes of curves on the hodograph are cylinders, with rulings parallel to the x^3-axis. These are Busemann's *cylindrical* flows. For $\mu^\alpha = u_\alpha$ (3.2) and (3.4) define the familiar Legendre transformation from the physical to the hodograph plane for plane flow, and (5.13) takes the usual form

$$(a^2\delta_{\alpha\beta} - u_\alpha u_\beta)(-1)^{\alpha+\beta}\, \partial^2 k/\partial u_{\alpha+1}\, \partial u_{\beta+1} = 0. \tag{5.31}$$

Finally, suppose $\det \| b_{\alpha\beta} \| \neq 0$, but $b_{\alpha\beta}$ and $k_{,\alpha\beta}$ are linearly dependent. By (5.9) and (5.14), for $u_\alpha = \mu^\alpha$, $\partial^2 u_3/\partial u_\alpha\, \partial u_\beta$ and $\partial^2 k/\partial\, u_\alpha\, \partial u_\beta$ are also linearly dependent. Hence for some $f(u_1\,, u_2)$

$$\partial^2 k/\partial u_\alpha\, \partial u_\beta = f\, \partial^2 u_3/\partial u_\alpha\, \partial u_\beta\,. \tag{5.32}$$

By (5.32)$(\partial^2 u_3/\partial u_\alpha\, \partial u_\beta)(\partial f/\partial u_\gamma) = (\partial^2 u_3/\partial u_\alpha \partial u_\gamma)(\partial f/\partial u_\beta)$. Since $\det \| \partial^2 u_3/\partial u_\alpha\, \partial u_\beta \| \neq 0$, then $\partial f/\partial u_\beta = 0$. (5.32) yields

$$k = B - A^i u_i\,, \tag{5.33}$$

where A^i and B are constants. By (3.5) and (5.33) all prototype lines pass through $x^i = A^i$. For each streamline S pass straight lines through $x^i = A^i$ and each point of S. The cone so constructed will be a stream sheet covered by streamlines similar to S. Accordingly such flows are said to be *conical,* a type considered by Busemann. The most familiar example is Taylor-Maccoll flow.

THEOREM 5.4: *If for a double wave* $\det \| cb_{\alpha\beta} + dk_{,\alpha\beta} \| = 0$ *for all c and d, or if* $b_{\alpha\beta}$ *and* $k_{,\alpha\beta}$ *are linearly dependent:*

(1) *The flow is conical or cylindrical.*
(2) *The prototype of any curve on the hodograph is developable.*
(3) *Conclusions* (2) *to* (4) *of Theorem* 5.3 *apply to these flows.*

To every double wave that is neither cylindrical nor conical there corresponds a conical flow with the same hodograph. Such general double waves will be called *skewed conical flows.*

Reconsider the conditions for hyperbolic, parabolic, or elliptic type for (5.12) and (5.15). They are $\sin \chi >, =, < 1/M$, where χ is the angle between the velocity and the direction of a prototype line. For subsonic flow (5.12) and (5.15) must be elliptic. In sharp contrast with plane flow, they need not be hyperbolic for supersonic double waves. To see this, consider a supersonic cylindrical flow based on a subsonic plane flow. Then (5.15) or (5.31) is elliptic.

6. Double waves with axisymmetric hodographs. An important class of examples can be constructed as follows. Assume $u_i = u_i(\mu)$ is axisymmetric about the u_3-axis. The hodograph may be represented by

$$u_1 = u(t)\cos\theta, \qquad u_2 = u(t)\sin\theta, \qquad u_3 = w(t) \tag{6.1}$$

for some $u(t)$ and $w(t)$. If (6.1) is a curve, two possibilities arise. If $u = 0$, (4.5) implies $w^2 = q^2 = a^2 = (\gamma - 1)c^2/(\gamma + 1)$, so (6.1) reduces to two points. If u and w are constant, (4.5) implies $a^2 = 0$, i.e. (6.1) is the circle $q = c$, $w =$ constant. This is a singular case of a velocity field with constant speed $q = c$ in a vacuum.

Next, suppose (6.1) is two-dimensional. Let $\mu^1 = t$, $\mu^2 = \theta$. Then by (5.8) and (5.9)

$$g_{11} = u'^2 + w'^2, \qquad g_{12} = 0, \qquad g_{22} = u^2,$$

$$b_{11} = (u'w'' - u''w')(u'^2 + w'^2)^{-1/2}, \tag{6.2}$$

$$b_{12} = 0, \qquad b_{22} = uw'(u'^2 + w'^2)^{-1/2}$$

where primes denote derivatives with respect to t. (5.12) implies

$$a^2u(w''u' - w'u'') + w'[a^2(u'^2 + w'^2) - (uu' + ww')^2] = 0. \tag{6.3}$$

This has the singular solution $q = c$, i.e. a spherical hodograph. It also has the solution $w = \text{constant}$, i.e. the hodograph lies in a plane and the corresponding flow is cylindrical. Hereafter, suppose w is not constant. With no loss of generality, set $t = w$. Then (6.3) becomes

$$a^2(uu'' - u'^2 - 1) + (uu' + w)^2 = 0. \tag{6.4}$$

This is a form of the differential equation for the hodographs of axisymmetric conical flows, of which Taylor-Maccoll flow or a convergent flow considered by Busemann [3] are particular examples. As stated at the end of Sec. 5, to these flows there correspond skewed conical flows with the same hodographs. To construct examples, find the function $k(w, \theta)$. By (5.14) and (6.2)

$$k_{,11} = \partial^2 k/\partial w^2 - \tfrac{1}{2}[\log (1 + u'^2)]\partial k/\partial w,$$

$$k_{,22} = \partial^2 k/\partial \theta^2 + [uu'/(1 + u'^2)]\partial k/\partial w.$$

By (5.12), (5.15), and (6.2) $b_{11}k_{,22} - b_{22}k_{,11} = 0$, so

$$\partial^2 k/\partial \theta^2 + (u/u'')\partial^2 k/\partial w^2 = 0. \tag{6.5}$$

A skewed Taylor-Maccoll flow can be constructed, *in the small*, by solving the ordinary differential equation (6.4), the *linear* partial differential equation (6.5), and finding the prototype lines (3.5). Thus, by relatively elementary processes a class of three-dimensional solutions of the non-linearized equation (2.10) can be constructed.

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ON TORSION OF PRISMS WITH LONGITUDINAL HOLES*

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Abstract. This paper presents a method of solution, called the method of images, for the torsion of prisms having one or more longitudinal holes. The method is applicable to prisms of the following four, and only four, sections: a rectangle, an equilateral triangle, an isosceles triangle and a 30°-60°-90° triangle. These four sections form a group by themselves.

The solution is obtained by adding to the known solution of a corresponding solid prism without holes a system of harmonic functions which vanish on the entire external boundary of the given section, and besides possess a singularity at the centre of each hole. Such a system of functions may be constructed from Weierstrass' Sigma function and its allied functions.

The solution is illustrated by applying it in detail to a rectangular prism having a central longitudinal hole. Numerical results are shown for the special case of a square prism.

Introduction. The torsion of a circular cylinder having longitudinal circular holes, with or without a central hole, has been investigated by Kondo[1] and by the present writer.[2] In the present paper, the investigation will be extended to a *prism*, which is also pierced by such longitudinal holes.

Both problems in fact belong to the same general class of torsion problems dealing with cylinders of *multi-connected* sections. Analytic solutions of such problems are generally difficult except in some simple cases, and indeed very few solutions have ever been found. It appears, however, that certain prisms of this nature can be solved by adapting to them the *method of images*. There are altogether four such prisms, the cross sections of which are as follows:

(1) a rectangle, including square as a special case,
(2) an equilateral triangle,
(3) an isosceles right triangle,
(4) a 30°-60°-90° triangle.

It is not difficult to show that by reflection about the edges each of the above four sections forms a doubly infinite set of images. Furthermore, it can be shown conclusively by theory of groups[3] that these four sections are the only ones which form such images.

The images formed in each case are shown in Figs. 1-4 respectively. In each figure the shaded area represents the *fundamental* region, or the given section of the prism;

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*Received August 9, 1950.

[1]M. Kondo, *The stresses in twisted circular cylinder having circular holes*, Phil. Mag. (7) **22**, 1089-1108 (1936).

[2]C. B. Ling, *Torsion of a circular tube with longitudinal circular holes*, Q. Appl. Math., **5**, 168-181 (1947).

[3]W. Burnside, *Theory of groups*, 2nd ed., Cambridge University Press, 1911, 410-418.

The regions marked by positive signs represent the images which are formed by an even number of reflections, while the regions marked by negative signs represent those which are formed by an odd number of reflections. Note that any two adjacent regions must have alternate positive and negative signs. The set of points in each figure represents the images due to a given point in the fundamental region.

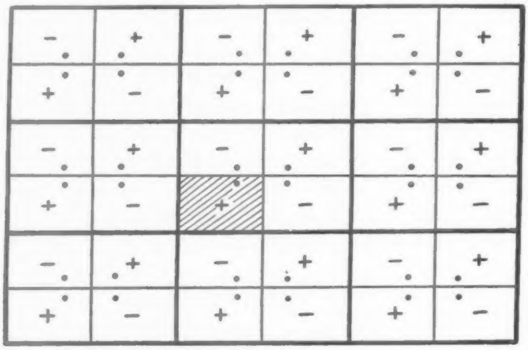

FIG. 1. Rectangle.

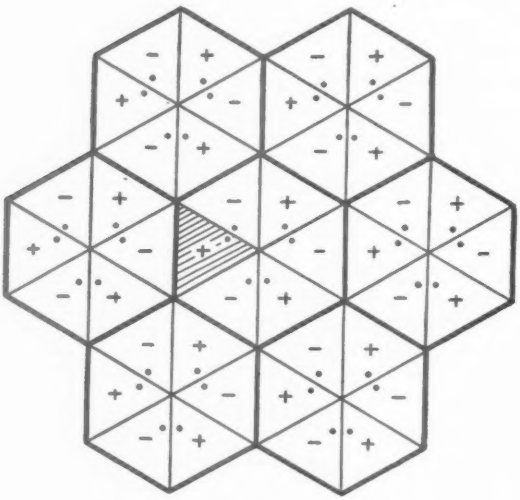

FIG. 2. Equilateral triangle.

It may be noted that in Fig. 1 the rectangles may be regarded as grouping themselves into *identical* rectangles each composed of four adjacent rectangles. Similarly, in Figs. 2, 3 and 4 the triangles may be regarded as grouping themselves into identical regular hexagons each composed of six adjacent triangles, into identical squares each composed of eight adjacent triangles, and into identical regular hexagons each composed of twelve adjacent triangles respectively. Such identical regions, which are indi-

cated by heavy lines, form a doubly infinite set in each case. Obviously, such groupings are not *unique*, but they are immaterial in the present treatment.

Whereas the present solution is restricted to the four sections as mentioned above,

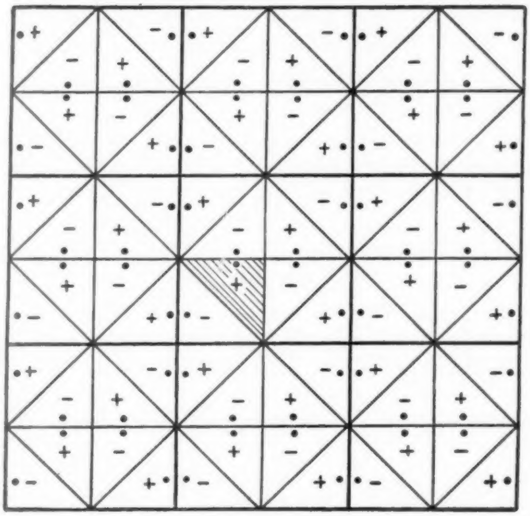

FIG. 3. Isosceles right triangle.

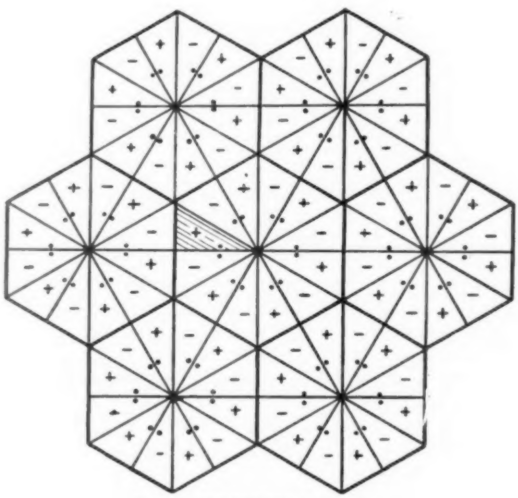

FIG. 4. 30°-60°-90° triangle.

there is however no further restriction theoretically as to the manner of distribution of the holes within the section. Naturally, the simplest is the case of a *single* circular hole, especially when the hole is symmetrically located within the section. The next to the

simplest is a group of *similar* circular holes. The latter implies that all the holes are of equal radii and symmetrically located within the section so that the properties of all the holes are alike, and consequently if the boundary conditions on any one of the holes are satisfied, the boundary conditions on all the other holes will be automatically satisfied. For convenience, the former will be referred to as the *one-hole* case and the latter as the *invariant* case. Various invariant cases are shown in Fig. 5. For the 30°-60°-90°

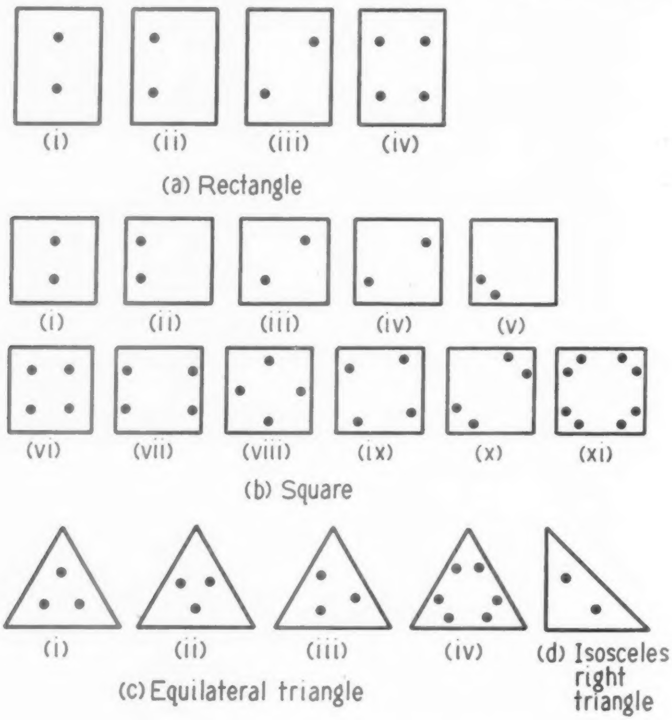

FIG. 5. Invariant cases (of more than one hole).

triangle, it appears that no such invariant distribution (or more than one hole) is possible.

Method of solution. In general the solution of the torsion problem requires a harmonic function, say ψ, whose conjugate function is single-valued, such that the function Ψ defined by the relation

$$\Psi = \psi - \tfrac{1}{2}(x^2 + y^2) \tag{1}$$

becomes a constant on each boundary of the given section; x and y being the rectangular coordinates in the plane of the section. As no generality is lost, the particular constant on the external boundary will henceforth be taken as zero.

Suppose that the function ψ is composed of two parts as follows:

$$\psi = \psi_0 + \psi_1 \tag{2}$$

where ψ_0 represents the solution of a corresponding solid section without holes. This implies that ψ_0 is a harmonic function which possesses a single-valued conjugate function and is equal to $\frac{1}{2}(x^2 + y^2)$ on the external boundary of the section. Consequently, the function ψ_1 must possess the following properties:

(1) it is also a harmonic function,
(2) its conjugate function is also single-valued,
(3) it vanishes on the entire external boundary.

Now, the functions ψ_0 for a solid prism of the four particular sections mentioned above are well known.[4] In particular, a solution has been given in a unified manner by Hay.[5] His method of solution is also described as a method of images, but is essentially based on a different consideration.

The present problem is thus reduced to find a function ψ_1 which possesses these properties, or more precisely, to find a complete system of such functions for ψ_1 so that the group of parametric coefficients attached to them can be further adjusted to satisfy the remaining boundary conditions at the internal boundaries of the given section. In addition, since the function ψ_0 for a solid section possesses no singularity inside the boundary, therefore the function ψ_1 must be of a different system such that it possesses

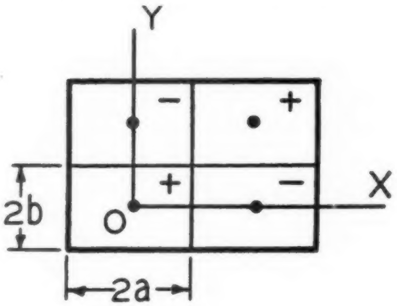

Fig. 6.

singularities inside the external boundary. Such singularities will eventually be excluded from the material of the section by the internal boundaries or holes. Thus, if the holes are circular, they are preferably placed at the centres of the holes. Naturally, the system of functions will be considerably simplified whenever the single hole or the group of holes are symmetrically located within the given section.

It appears that the system of functions for the four sections mentioned above may be constructed from Weierstrass' Sigma function and its allied functions. However, as the sections are so different in nature, it seems that no general expression can be put forward for all the four sections as a whole. In the following, the case of a rectangular prism having a central longitudinal circular hole will be chosen as an illustration. The system

[4]St. Venant, Mém. des Savants étrangers **14**, 233-560 (1856), for rectangle and equilateral triangle.
B. G. Galerkin, Bull. de l'Acad. des Sci. de Russie **13**, 111-118 (1919) for isosceles right triangle.
G. Kolossoff, Comptes Rendus **178**, 2057-2060 (1924), for isosceles right triangle.
B. R. Seth, Q. J. of Math. **5**, 161-171 (1934), for equilateral triangle, isosceles right triangle and 30°-60°-90° triangle.

[5]G. E. Hay, Proc. of London Math. Soc. **45**, 382-397 (1939).

of functions will first be constructed. Subsequently, formulas for shear stresses and torsional modulus will be derived. Methods will also be given for evaluating the coefficients or functions involved in the solution. Finally, numerical results will be shown for the special case of a square prism.

Rectangular prism having a central longitudinal hole. A class of doubly periodic functions. Consider in the z-plane an identical rectangle composed of four adjacent rectangles, including the fundamental rectangle, as shown previously in Fig. 1. Let the dimensions of each rectangle be equal to $2a \times 2b$ and the origin of the coordinates be at the centre of the fundamental rectangle as shown in Fig. 6. Now consider the doubly infinite set of image points due to a point at the origin. Since the identical rectangle is of double periods $4a$ and $4ib$, the affixes of the set of image points in the positive regions, including the given point in the fundamental rectangle, are given by

$$\left.\begin{aligned} P_{mn} &= 4ma + 4nib \\ \text{and}\qquad Q_{mn} &= 2(2m+1)a + 2(2n+1)ib = P_{mn} + 2a + 2ib, \end{aligned}\right\} \tag{3}$$

where m and n are extended to all positive and negative integers including zero. Again, the affixes of the set of image points in the negative regions are given by

$$\left.\begin{aligned} P^*_{mn} &= 2(2m+1)a + 4nib = P_{mn} + 2a \\ \text{and}\qquad Q^*_{mn} &= 4ma + 2(2n+1)ib = P_{mn} + 2ib, \end{aligned}\right\} \tag{4}$$

where m and n are extended to the same range of values as before.

A function with a logarithmic singularity at each such point will be defined by

$$W_0(z) = -\log \frac{\sigma_1(z)\sigma_2(z)}{\sigma^*_1(z)\sigma^*_2(z)}, \tag{5}$$

where σ_1, σ_2 and σ^*_1, σ^*_2 are functions of z defined by the following doubly infinite products:

$$\left.\begin{aligned} \sigma_1(z) &= z \prod_{m,n=-\infty}^{\infty}{}' \left(1 - \frac{z}{P_{mn}}\right) \exp\left(\frac{z}{P_{mn}} + \frac{z^2}{2P^2_{mn}}\right), \\ \sigma_2(z) &= \prod_{m,n=-\infty}^{\infty} \left(1 - \frac{z}{Q_{mn}}\right) \exp\left(\frac{z}{Q_{mn}} + \frac{z^2}{2Q^2_{mn}}\right), \\ \sigma^*_1(z) &= \prod_{m,n=-\infty}^{\infty} \left(1 - \frac{z}{P^*_{mn}}\right) \exp\left(\frac{z}{P^*_{mn}} + \frac{z^2}{2P^{*2}_{mn}}\right), \\ \sigma^*_2(z) &= \prod_{m,n=-\infty}^{\infty} \left(1 - \frac{z}{Q^*_{mn}}\right) \exp\left(\frac{z}{Q^*_{mn}} + \frac{z^2}{2Q^{*2}_{mn}}\right), \end{aligned}\right\} \tag{6}$$

in each of which the double multiplication is extended to all positive and negative integers of m and n including zero, except that in the first the pair of simultaneous zeros is omitted. This omission is distinguished by adding to the product sign an accent as indicated. It may be noted that the function σ_1 is Weierstrass' Sigma function while the remaining three functions[6] are closely allied to σ_1. Thus, we have

[6]Cf. E. T. Copson, *Functions of a complex variable*, Oxford University Press, 1935, p. 378.

$$
\begin{aligned}
W_0(z) = -\log z &- \sum_{m,n=-\infty}^{\infty}{}' \left\{\log\left(1 - \frac{z}{P_{mn}}\right) + \frac{z}{P_{mn}} + \frac{z^2}{2P_{mn}^2}\right\} \\
&- \sum_{m,n=-\infty}^{\infty} \left\{\log\left(1 - \frac{z}{Q_{mn}}\right) + \frac{z}{Q_{mn}} + \frac{z^2}{2Q_{mn}^2}\right\} \\
&+ \sum_{m,n=-\infty}^{\infty} \left\{\log\left(1 - \frac{z}{P_{mn}^*}\right) + \frac{z}{P_{mn}^*} + \frac{z^2}{2P_{mn}^{*2}}\right\} \\
&+ \sum_{m,n=-\infty}^{\infty} \left\{\log\left(1 - \frac{z}{Q_{mn}^*}\right) + \frac{z}{Q_{mn}^*} + \frac{z^2}{2Q_{mn}^{*2}}\right\},
\end{aligned}
\tag{7}
$$

where the accent on the summation sign indicates likewise the omission of the pair of simultaneous zeros of m and n from the double summation. By expanding the logarithmic terms in the neighborhood of the origin, this leads to

$$
W_0(z) = -\log z - \sum_{k=3}^{\infty} \frac{1}{k}\, \Omega_k z^k, \tag{8}
$$

where, for k W 3,

$$
\Omega_k = -\sum_{m,n=-\infty}^{\infty}{}' \frac{1}{P_{mn}^k} + \sum_{m,n=-\infty}^{\infty} \left(\frac{1}{P_{mn}^{*k}} + \frac{1}{Q_{mn}^{*k}} - \frac{1}{Q_{mn}^k}\right). \tag{9}
$$

It is readily shown by symmetry that the coefficient Ω_k vanishes identically when k is odd, and is real when k is even.

A class of analytic functions with poles of integral orders at the foregoing sets of image points will be defined by

$$
W_s(z) = \frac{(-1)^s}{(s-1)!} \frac{d^s}{dz^s} W_0(z), \tag{10}
$$

where s is the order of poles of the function. When $s \geq 2$, the class of functions are doubly periodic or elliptic functions.

The form of the functions thus derived is different according as s is odd or even. The results are as follows. The initial function W_0 is also rewritten for the sake of uniformity.

and

$$
\left.
\begin{aligned}
W_0(z) &= -\log z - \sum_{n=2}^{\infty} {}^{2n}\alpha_0 z^{2n}, \\
W_2(z) &= \frac{1}{z^2} - \sum_{n=1}^{\infty} {}^{2n}\alpha_2 z^{2n}, \\
W_{2s}(z) &= \frac{1}{z^{2s}} - \sum_{n=0}^{\infty} {}^{2n}\alpha_{2s} z^{2n}, \qquad (s \geq 2) \\
W_{2s+1}(z) &= \frac{1}{z^{2s+1}} + \sum_{n=0}^{\infty} {}^{2n+1}\alpha_{2s+1} z^{2n+1}, \qquad (s \geq 0)
\end{aligned}
\right\}
\tag{11}
$$

where

$$\left.\begin{aligned} {}^{n}\alpha_{s} &= \binom{n+s-1}{n}\Omega_{n+s} \\ {}^{n}\alpha_{0} &= \frac{1}{n}\,\Omega_{n}\,, \qquad {}^{1}\alpha_{1} = 0. \end{aligned}\right\} \tag{12}$$

and in particular

Note that here the coefficient ${}^{0}\alpha_{2}$ is not defined. It will be reserved for later usage.

Now split these functions into real and imaginary parts as follows: For $s \geq 0$,

$$W_s(z) = S_s(x, y) - iT_s(x, y). \tag{13}$$

Also, define a pair of polar coordinates (r, θ) by

$$z = x + iy = re^{i\theta}. \tag{14}$$

We then have, since ${}^{n}\alpha_{s}$ is real,

$$\left.\begin{aligned} S_0 &= -\log r - \sum_{n=2}^{\infty} {}^{2n}\alpha_0 r^{2n} \cos 2n\theta, \\ T_0 &= \theta + \sum_{n=2}^{\infty} {}^{2n}\alpha_0 r^{2n} \sin 2n\theta, \\ S_2 &= \frac{\cos 2\theta}{r^2} - \sum_{n=1}^{\infty} {}^{2n}\alpha_2 r^{2n} \cos 2n\theta, \\ S_{2s} &= \frac{\cos 2s\theta}{r^{2s}} - \sum_{n=0}^{\infty} {}^{2n}\alpha_{2s} r^{2n} \cos 2n\theta, \qquad (s \geq 2) \\ T_{2s} &= \frac{\sin 2s\theta}{r^{2s}} + \sum_{n=1}^{\infty} {}^{2n}\alpha_{2s} r^{2n} \sin 2n\theta, \qquad (s \geq 1) \\ S_{2s+1} &= \frac{\cos (2s+1)\theta}{r^{2s+1}} + \sum_{n=0}^{\infty} {}^{2n+1}\alpha_{2s+1} r^{2n+1} \cos (2n+1)\theta, \qquad (s \geq 0) \\ T_{2s+1} &= \frac{\sin (2s+1)\theta}{r^{2s+1}} - \sum_{n=0}^{\infty} {}^{2n+1}\alpha_{2s+1} r^{2n+1} \sin (2n+1)\theta, \qquad (s \geq 0) \end{aligned}\right\} \tag{15}$$

The preceding expressions give the expansions of the harmonic functions S_s and T_s in the neighborhood of the origin. Note that S_{2s} is even in both x and y, T_{2s} is odd in both x and y, S_{2s+1} is odd in x but even in y, and T_{2s+1} is even in x but odd in y.

It can be shown from Cauchy-Riemann differential equations that the following relations exist.

$$\left.\begin{aligned} \frac{\partial S_0}{\partial x} &= -\frac{\partial T_0}{\partial y} = -S_1\,, \qquad \frac{\partial T_0}{\partial x} = \frac{\partial S_0}{\partial y} = -T_1\,, \\ \frac{\partial S_s}{\partial x} &= -\frac{\partial T_s}{\partial y} = -sS_{s+1}\,, \qquad \frac{\partial T_s}{\partial x} = \frac{\partial S_s}{\partial y} = -sT_{s+1}\,. \end{aligned}\right\} \tag{16}$$

Furthermore, it can be shown that for $s \geq 1$:

$$\begin{aligned}
&(1)\quad \text{when } x = 0, \quad \pm 2a, \quad \pm 4a, \cdots, \\
&\qquad\qquad T_{2s} = 0, \qquad S_{2s+1} = 0; \\
&(2)\quad \text{when } x = \pm a, \quad \pm 3a, \quad \pm 5a, \cdots, \\
&\qquad\qquad S_2 = \text{const.}, \qquad S_{2s} = 0, \qquad T_{2s+1} = 0; \\
&(3)\quad \text{when } y = 0, \quad \pm 2b, \quad \pm 4b, \cdots, \\
&\qquad\qquad T_{2s} = 0, \qquad T_{2s+1} = 0; \\
&(4)\quad \text{when } y = \pm b, \quad \pm 3b, \quad \pm 5b, \cdots, \\
&\qquad\qquad S_2 = \text{const.}, \qquad S_{2s} = 0, \qquad S_{2s+1} = 0.
\end{aligned} \tag{17}$$

The two *real* constants for S_2 are in fact identical and will be denoted by ${}^0\alpha_2$. It is found that

$${}^0\alpha_2 = 2\wp(2a + 2bi), \tag{18}$$

where $\wp(z)$ is Weisrstrass' elliptic function of double periods $4a$ and $4bi$. It is seen that the functions S_{2s} all vanish on the external boundary of the given rectangle, i.e., at $x = \pm a$ and $y = \pm b$, except S_2 which becomes a constant. Therefore, the system of functions is useful in constructing the function ψ_1.

The solution. For a solid rectangular prism of cross section $2a \times 2b$, the function ψ_0 is known as[7]

$$\begin{aligned}
\psi_0 &= b^2 + \frac{1}{2}(x^2 - y^2) \\
&\quad - 4b^2\left(\frac{2}{\pi}\right)^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \frac{\cosh (2n+1)\pi x/2b}{\cosh (2n+1)\pi a/2b} \cos \frac{(2n+1)\pi y}{2b},
\end{aligned} \tag{19}$$

the origin being at the centre of the rectangular section. With the aid of the expansion

$$\cosh px \cos py = \sum_{m=0}^{\infty} \frac{1}{(2m)!} (pr)^{2m} \cos 2m\theta \tag{20}$$

the above function may be expressed in polar coordinates as follows:

$$\psi_0 = f_0 + \sum_{n=1}^{\infty} f_{2n} r^{2n} \cos 2n\theta \tag{21}$$

where, for $n \geq 0$,

$$f_{2n} = b^2 \delta_{0,n} + \frac{1}{2}\delta_{1,n} - \frac{4}{b(2n)!} \sum_{m=0}^{\infty} (-1)^m \left\{\frac{(2m+1)\pi}{2b}\right\}^{2n-3} \operatorname{sech} \frac{(2m+1)\pi a}{2b} \tag{22}$$

in which $\delta mn = 1$ or 0, according as $m = n$, or $m \neq n$.

Now, we construct

[7] A. E. H. Love, *Mathematical theory of elasticity*, 4th ed., Dover Publications, 1944, 317-318. Note that the constant is modified so that $\psi_0 = \frac{1}{2}(x^2 + y^2)$ on the boundary.

$$\psi_1 = \sum_{s=1}^{\infty} A_{2s} S_{2s}(x, y) - {}^0\alpha_2 A_2 \,, \tag{23}$$

where A_{2s} are arbitrary constants to be determined. The initial function S_0 is rejected on the ground that its conjugate function is not single-valued. The function ψ_1 thus constructed evidently meets all the requirements as outlined previously.

To adjust the remaining condition on the internal boundary or the central hole, we have in terms of polar coordinates,

$$\begin{aligned}\Psi &= -\frac{1}{2}(x^2 + y^2) + \psi_0 + \psi_1 \\ &= -\frac{1}{2}r^2 + f_0 - \sum_{s=1}^{\infty} {}^0\alpha_{2s} A_{2s} \\ &\quad + \sum_{n=1}^{\infty} \left(f_{2n} r^{2n} + A_{2n} r^{-2n} - r^{2n} \sum_{s=1}^{\infty} {}^{2n}\alpha_{2s} A_{2s} \right) \cos 2n\theta.\end{aligned} \tag{24}$$

Hence the boundary condition on the rim of the circular hole, where $r = \lambda$ say, is satisfied provided that for $n \geq 1$,

$$f_{2n}\lambda^{2n} + A_{2n}\lambda^{-2n} - \lambda^{2n} \sum_{s=1}^{\infty} {}^{2n}\alpha_{2s} A_{2s} = 0$$

or

$$A_{2n} = -f_{2n}\lambda^{4n} + \lambda^{4n} \sum_{s=1}^{\infty} {}^{2n}\alpha_{2s} A_{2s} \,. \tag{25}$$

The value of Ψ on the rim of hole then becomes a constant, say Ψ_0, as follows:

$$\Psi_0 = -\frac{1}{2}\lambda^2 + f_0 - \sum_{s=1}^{\infty} {}^0\alpha_{2s} A_{2s} \,. \tag{26}$$

The system of linear equations in (25) may be solved by successive approximations as follows. Write

$$A_{2n} = \sum_{p=0}^{\infty} A_{2n}^{(p)} \tag{27}$$

where

$$A_{2n}^{(0)} = -f_{2n}\lambda^{4n}$$

and, by iteration,

$$A_{2n}^{(p)} = \lambda^{4n} \sum_{s=1}^{\infty} {}^{2n}\alpha_{2s} A_{2s}^{(p-1)}. \tag{28}$$

Naturally, the validity of the solution depends upon the convergence of the series (27). From physical considerations alone, it seems likely that there will be convergence as long as the rim of hole does not touch the external boundary, i.e., when

$$\lambda < \min (a, b). \tag{29}$$

Torsional modulus. The torsional modulus[8] of the prism is given by

$$H = 2 \iint \Psi \, dx \, dy + 2\pi\lambda^2\Psi_0 \,, \tag{30}$$

[8] R. V. Southwell, *Theory of elasticity*, 2nd ed., Oxford University Press, 1941, p. 323.

where the double integration is extended over the entire rectangular section, excluding the hole.

There is no difficulty in evaluating the following integrals:

$$\begin{aligned}\iint \psi_0\, dx\, dy &= 4\int_0^b\int_0^a \psi_0\, dx\, dy - 2\int_0^\lambda\int_0^\pi \psi_0 r\, d\theta\, dr\\ &= \frac{2}{3}\, ab(a^2+5b^2) - \pi\lambda^2 f_0 - \frac{1}{2}\left(\frac{4}{\pi}\right)^5 b^4 \sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\tanh\frac{(2n+1)\pi a}{2b},\end{aligned} \tag{31}$$

$$\begin{aligned}\iint (x^2+y^2)\, dx\, dy &= 4\int_0^b\int_0^a (x^2+y^2)\, dx\, dy - 2\int_0^\lambda\int_0^\pi r^3\, d\theta\, dr\\ &= \frac{4}{3}\, ab(a^2+b^2) - \frac{1}{2}\pi\lambda^4.\end{aligned} \tag{32}$$

But in evaluating the integral $\iint \psi_1\, dx\, dy$ the foregoing method fails owing to the fact that ψ_1 possesses a singularity at the origin. However, it may be evaluated by means

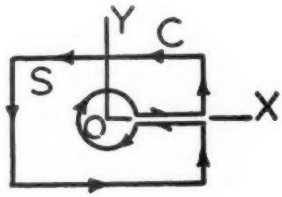

FIG. 7.

of contour integration based upon a corollary from Green's theorem. Suppose that F is a function of z regular in a domain S which is enclosed by a contour C. Then we find

$$\iint_S \frac{dF}{dz}\, dS = \frac{1}{2}\, i\int_C F\, d\bar{z}, \tag{33}$$

where $\bar{z}$ is the conjugate of z; the contour being taken in a counter-clockwise direction: Now, put $F = W_s(z)$ where $s \geq 1$. Then

$$\iint_S W_{s+1}(z)\, dS = -\frac{i}{2s}\int_C W_s(z)\, d\bar{z}. \tag{34}$$

Thus by referring to the contour C in Fig. 7, for $s \geq 2$,

$$\begin{aligned}\iint S_{2s}\, dx\, dy &= R.P.\iint_S W_{2s}(z)\, dS = -\frac{1}{2(2s-1)}\, R.P.\int_C iW_{2s-1}(z)\, d\bar{z}\\ &= -\frac{4}{(2s-1)(2s-2)}\, T_{2s-2}(a, b) + \pi\Omega_{2s}\lambda^2.\end{aligned} \tag{35}$$

In particular, for $s = 1$,

$$\iint S_2 \, dx \, dy = -\frac{1}{2} R.P. \int_C iW_1(z) \, d\bar{z} = -4T_0(a, b). \tag{36}$$

Consequently, we have

$$\begin{aligned} \iint \psi_1 \, dx \, dy = & -A_2\{4T_0(a, b) + (4ab - \pi\lambda^2)^0\alpha_2\} \\ & - \sum_{s=2}^{\infty} A_{2s}\left\{\frac{4}{(2s-1)(2s-2)} T_{2s-2}(a, b) - \pi\Omega_{2s}\lambda^2\right\}. \end{aligned} \tag{37}$$

Hence the torsional modulus is equal to

$$\begin{aligned} H = & \frac{16}{3} ab^3 - \frac{1}{2}\pi\lambda^4 - \left(\frac{4}{\pi}\right)^5 b^4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^5} \tanh \frac{(2n+1)\pi a}{2b} \\ & - 2A_2\{4T_0(a, b) + (4ab - \pi\lambda^2)^0\alpha_2\} - 8 \sum_{s=1}^{\infty} \frac{1}{2s(2s+1)} A_{2s+2}T_{2s}(a, b). \end{aligned} \tag{38}$$

The resulting twisting couple is given by

$$T = \mu\tau H \tag{39}$$

where μ is the modulus of rigidity of the material and τ is the angle of twist per unit length of the prism.

Stress components. The non-vanishing stress components of the prism are two shear stress components given in terms of rectangular coordinates by

$$Z_x = \mu\tau \frac{\partial\Psi}{\partial y}, \qquad Z_y = -\mu\tau \frac{\partial\Psi}{\partial x} \tag{40}$$

or in terms of polar coordinates by

$$Z_r = \frac{\mu\tau}{r} \frac{\partial\Psi}{\partial\theta}, \qquad Z_\theta = -\mu\tau \frac{\partial\Psi}{\partial r} \tag{41}$$

It is now a straightforward matter to calculate the stress at any point in the prism. In particular the shear stresses on the external and internal boundaries are as follows:

$$\left.\begin{aligned} [Z_x]_{y=b} &= -2\mu\tau\left\{b - \frac{8b}{\pi^2} \sum_{n=0}^{\infty} \frac{\cosh (2n+1)\pi x/2b}{(2n+1)^2 \cosh (2n+1)\pi a/2b} + \sum_{s=1}^{\infty} sA_{2s}T_{2s+1}(x, b)\right\} \\ [Z_y]_{x=a} &= 2\mu\tau\left\{\frac{8b}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \tanh \frac{(2n+1)\pi a}{2b} \cos \frac{(2n+1)\pi y}{2b} \right. \\ & \qquad \left. + \sum_{s=1}^{\infty} sA_{2s}S_{2s+1}(a, y)\right\} \\ [Z_\theta]_{r=\lambda} &= \mu\tau\left(\lambda + 4 \sum_{n=1}^{\infty} \frac{nAn}{\lambda^{2n+1}} \cos 2n\theta\right) \end{aligned}\right\} \tag{42}$$

The maximum shear stress occurs on the boundary at those points which are nearest to the next boundary. Due to the presence of the holes, the greatest shear stress does not necessarily occur on the external boundary.

Evaluation of coefficients. Numerical results will be given for the case of a square prism of cross section $2a \times 2a$ (i.e., $a = b$) having a central hole of two different radii $\lambda = a/2$ and $a/3$ respectively.

When $a = b$, it can be further shown that for $k \geq 1$,

$$\Omega_{4k+2} = 0,$$
$$\Omega_{4k} = \frac{1}{(2a)^{4k}}\left(1 - \frac{2(-1)^k}{2^{2k}}\right)\sigma_{4k}\,, \tag{43}$$

where

$$\sigma_{4k} = \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m+ni)^{4k}}\,. \tag{44}$$

There exists an algebraic relation between the coefficients[9] σ_{4k}. For $k \geq 2$,

$$\frac{1}{3}(2k-3)(4k+1)C_{4k} = C_4C_{4k-4} + C_8C_{4k-8} + C_{12}C_{4k-12} + \cdots + C_{4k-4}C_4 \tag{45}$$

where, for $k \geq 1$,

$$C_{4k} = (4k-1)\sigma_{4k} \tag{46}$$

The values of σ_{4k} have been computed by the present writer some time ago.[10] They

TABLE I. COEFFICIENTS FOR A SQUARE PRISM

n	σ_{4n}	$\Omega_{4n}(2a)^{4n}$	$f_{2n}a^{2n-2}$	$\frac{A_{2n}}{a^{2n+2}}$ for $\lambda = \frac{a}{2}$	$\frac{A_{2n}}{a^{2n+2}}$ for $\lambda = \frac{a}{3}$
0	—	—	8.93704×10^{-2}	—	—
1	3.151212	4.726818	0	0	0
2	4.255773	3.723801	-9.11851×10^{-2}	3.56902×10^{-4}	1.38992×10^{-5}
3	3.938849	4.061938	0	0	0
4	4.015695	3.984322	1.92418×10^{-3}	-2.84698×10^{-8}	-4.46471×10^{-11}
5	3.996097	4.003902	0	0	0
6	4.000977	3.999023	-1.20581×10^{-4}	7.77531×10^{-12}	4.28303×10^{-16}
7	3.999756	4.000244	0	0	0
8	4.000061	3.999939	1.1841×10^{-5}	-2.4495×10^{-15}	-6.3624×10^{-21}
9	3.999985	4.000015	0	0	0
10	4.000004	3.999996	-1.4543×10^{-6}	1.4594×10^{-18}	1.2010×10^{-25}
11	3.999999	4.000001	0	0	0
12	4.000000	4.000000	2.0472×10^{-7}	-6.7229×10^{-22}	-2.5589×10^{-30}

are reproduced in Table I where values of Ω_{4k} and f_{4n} are also tabulated. It is noted that by symmetry, for $n \geq 0$,

$$f_{4n+2} = 0, \qquad A_{4n+2} = 0. \tag{47}$$

[9]Cf. E. T. Copson, *loc. cit.*, p. 360.

[10]Doctoral thesis by C. B. Ling presented to London University, England (1937).

Values of ${}^{4n}\alpha_{4s}$ are shown in Table II. With these values, the coefficients A_{4n} can now be computed from (27) by successive approximations. The results for two different radii $\lambda = a/2$ and $a/3$ are shown together in Table I.

TABLE II. ${}^{4n}\alpha_{4s}a^{4n+4s}$ FOR A SQUARE PRISM

$4s$	$4n = 4$	$4n = 8$	$4n = 12$	$4n = 16$	$4n = 20$	$4n = 24$
4	5.09114×10^{-1}	1.63628×10^{-1}	2.76622×10^{-2}	3.70005×10^{-3}	4.22136×10^{-4}	4.35886×10^{-5}
8	3.27256×10^{-1}	3.91222×10^{-1}	1.92402×10^{-1}	5.84357×10^{-2}	1.32335×10^{-2}	2.44895×10^{-3}
12	8.29864×10^{-2}	2.88604×10^{-1}	3.22282×10^{-1}	1.94292×10^{-1}	7.88560×10^{-2}	2.42858×10^{-2}
16	1.48002×10^{-2}	1.16871×10^{-1}	2.59056×10^{-1}	2.79896×10^{-1}	1.89056×10^{-1}	9.14617×10^{-2}
20	2.11068×10^{-3}	3.30837×10^{-2}	1.31427×10^{-1}	2.36320×10^{-1}	2.50741×10^{-1}	1.82006×10^{-1}
24	2.61531×10^{-4}	7.34684×10^{-3}	4.85717×10^{-2}	1.37193×10^{-1}	2.18408×10^{-1}	2.29133×10^{-1}

To proceed further in computing the torsional modulus and shear stresses on the boundaries, values of ${}^0\alpha_2$, $T_{4s+2}(a, a)$ and $T_{4s+1}(x, a)$ are required.

It is readily shown that when $a = b$,

$$ {}^0\alpha_2 = 0. \tag{53}$$

Now, define similarly a class of analytic functions $W_s^*(z)$ whose initial function is

$$W_0^*(z) = -\log \{\sigma_1(z)\sigma_2(z)\}, \tag{54}$$

where σ_1 and σ_2 are defined before in (6). The expansion of $W_s^*(z)$ in the neighborhood of the origin, for $s \geq 1$, is

$$W_s^*(z) = \frac{(-1)^s}{(s-1)!}\frac{d^s}{dz^s}W_0^*(z) = \frac{1}{z^s} + (-1)^s \sum_{n=0}^{\infty} {}^n\alpha_s^* z^n, \tag{55}$$

where, for $a = b$,

$$ {}^n\alpha_s^* = (-1)^{(n+s)/4}\binom{n+s-1}{n}\frac{\sigma_{n+s}}{(2/2a)^{n+s}}\,. \tag{56}$$

σ_{n+s} is defined in (44), which vanishes unless the suffix $(n + s)$ is an integral multiple of 4. An algebraic relation between the functions is as follows: For $s \geq 5$,

$$\frac{1}{6}(s-3)(s-2)B_s = B_2B_{s-2} + B_3B_{s-3} + B_4B_{s-4} + \cdots + B_{s-2}B_2 \tag{57}$$

where, for $s \geq 2$,

$$B_s = (s-1)W_s^*(z) = (s-1)\{S_s^*(x, y) - iT_s^*(x, y)\} \tag{58}$$

When $a = b$, it can be further shown that for $s \geq 1$,

$$\begin{aligned} W_{2s+1}^*(a + ai) &= 0, \\ W_{4s}^*(a + ai) &= S_{4s}^*(a, a), \\ W_{4s-2}^*(a + ai) &= -iT_{4s-2}^*(a, a) = -\tfrac{1}{2}iT_{4s-2}(a, a). \end{aligned} \tag{59}$$

The two initial functions may be found from

$$W_2^*(a + ai) = -\frac{i(15\sigma_4)^{1/2}}{8a^2}, \qquad W_4^*(a + ai) = -\frac{5\sigma_4}{32a^4} \cdot \tag{60}$$

With the aid of (57) for successive functions, values of $T_{4s+2}(a, a)$ can then be found without difficulty.

To evaluate $T_{4s+1}(x, a)$, it is noted that for $a = b$,

$$T_{4s+1}(x, a) = 2T_{4s+1}^*(x, a) = R.P.\{2iW_{4s+1}^*(x + ia)\} \tag{61}$$

In general, the three initial functions of $W_s^*(z)$ may be found from

$$W_2^*(z) = \left\{\frac{W_3^*(x) - iW_3^*(y)}{W_2^*(x) + W_2^*(y)}\right\}^2 - W_2^*(x) + W_2^*(y),$$
$$W_3^*(z) = \left[\{W_2^*(z)\}^3 + \frac{15\sigma_4}{64a^4} W_2^*(z)\right]^{1/2}, \tag{62}$$
$$W_4^*(z) = \{W_2^*(z)\}^2 + \frac{5\sigma_4}{64a^4} \cdot$$

With also the aid of (55) and (57), values of $T_{4s+1}(x, a)$ can then be computed for any particular value of x.

The results are tabulated in Tables III, IV and V.

TABLE III. $T_{4s-2}(a, a)$ and $T_{4s+1}(x, a)$ FOR A SQUARE PRISM

s	$T_{4s-2}(a, a)$	$T_{4s+1}(x, a)$					
		$x = 0$	$x = 0.2a$	$x = 0.4a$	$x = 0.6a$	$x = 0.8a$	$x = a$
1	1.71880	1.93656	0.46903	−0.25063	−0.43000	−0.25138	0
2	−0.50778	2.00239	−0.16957	−0.48920	0.04270	0.10612	0
3	0.12501						

TABLE IV. CROSS SECTIONAL AREA A AND TORSIONAL MODULUS H OF A SQUARE PRISM

λ/a	A/a^2	H/a^4	% reduction of A	% reduction of H
$\frac{1}{2}$	3.2146	2.1502	19.63	4.40
$\frac{1}{2}$	3.6509	2.2298	8.73	0.86
Solid	4.0000	2.2492	0	0

TABLE V. SHEAR STRESSES ON BOUNDARIES OF A SQUARE PRISM

$Z_x/\mu\tau a$ on $y = a$				$Z_\theta/\mu\tau a$ on $r = \lambda$		
x/a	$\lambda = \frac{1}{2}a$	$\lambda = \frac{1}{3}a$	Solid	$\theta°$	$\lambda = \frac{1}{2}a$	$\lambda = \frac{1}{3}a$
0	−1.353	−1.351	−1.351	0	0.524	0.360
0.2	−1.316	−1.315	−1.315	5	0.523	0.359
0.4	−1.210	−1.211	−1.211	10	0.518	0.354
0.6	−1.014	−1.014	−1.014	20	0.504	0.338
0.8	−0.683	−0.684	−0.684	30	0.488	0.320
1	0	0	0	45	0.476	0.306

The writer wishes to thank Mr. T. C. Lee of the Aeronautical Research Laboratory, Taiwan, for his assistance in preparing the manuscript.

THE TORSION AND STRETCHING OF SPIRAL RODS (I)*

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SYNOPSIS

In this paper the torsion or the stretching problem for a spiral rod is treated theoretically. The equations of equilibrium expressed in terms of displacements are reduced to forms which are independent of one co-ordinate. They are readily integrated for the particular case where the helix angle is small, and the corresponding displacements and stresses can be expressed in forms which contain three arbitrary plane harmonic functions, determination of which is dependent upon the shape of the section. As an application of the general solution, the problem for an elliptic section is solved explicitly.

Two-dimensional problems in elasticity have been studied extensively from early times on account of their simplicity in stress analysis and their useful applications in many engineering problems. For a similar reason, various problems of axially symmetrical stress distribution have been investigated by many writers.

In this paper we shall treat the torsion or the stretching problem for a spiral rod. The stress distribution for this case is neither two-dimensional nor axially symmetrical, and each stress does not vanish in general and consequently the analysis becomes somewhat complicated. But the problem is not a three dimensional one without any restriction, since if we rotate the co-ordinate axes about the axis of the helix so as to coincide with the fixed directions with respect to a section which is perpendicular to the axis of helix, then the stress distribution referred to the rotating axes is the same in any section.

Starting from the equations of equilibrium expressed in terms of displacements, we shall introduce equations which are independent of the position of the section. The differential equations of displacements are readily integrated for the particular case where the helix angle is small. The corresponding displacements and stresses are expressed in forms which contain three arbitrary plane harmonic functions, determination of which is dependent upon the shape of the section, and thus we can considerably simplify the problem.

We shall take the axis of the helix as the z-axis, and shall denote the displacements in the x, y, z directions by u, v and w, respectively. Then the equations of equilibrium can be expressed in the forms[1]

$$\begin{aligned}(\lambda + \mu)\frac{\partial \Delta}{\partial x} + \mu\nabla^2 u &= 0,\\ (\lambda + \mu)\frac{\partial \Delta}{\partial y} + \mu\nabla^2 v &= 0,\\ (\lambda + \mu)\frac{\partial \Delta}{\partial z} + \mu\nabla^2 w &= 0,\end{aligned} \tag{1}$$

where λ, μ are Lamé constants and Δ is the cubical dilatation.

*Received November 16, 1950.

[1]H. Love, *The mathematical theory of elasticity*, 4th ed., 1927, p. 133.

We shall make the following transformation

$$x' + iy' = e^{ikz}(x + iy), \tag{2}$$

where k is a constant which is related to the obliquity of the helix.

Let u', v' be the displacements in the x' and y' directions respectively. It follows that

$$u' + iv' = e^{ikz}(u + iv). \tag{3}$$

Let us put the displacements in the form

$$\begin{aligned} u' &= u_1(x', y') - \alpha y'z, \\ v' &= v_1(x', y') + \alpha x'z, \\ w &= w_1(x', y') + \beta z, \end{aligned} \tag{4}$$

where α, β are constants. Using the expressions for the displacements in Eq. (4), we obtain the cubical dilatation

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial u_1}{\partial x'} + \frac{\partial v_1}{\partial y'} - kD_2(w_1) + \beta,$$

and

$$\frac{\partial \Delta}{\partial x} = \frac{\partial \Delta}{\partial x'} \cos kz + \frac{\partial \Delta}{\partial y'} \sin kz.$$

Remembering the relation in Eq. (3), we have

$$\begin{aligned} \nabla^2 u = \{\nabla_1^2 u_1 + k^2 D_1(u_1) - 2k^2 D_2(v_1)\} \cos kz \\ + \{\nabla_1^2 v_1 + k^2 D_1(v_1) + 2k^2 D_2(u_1)\} \sin kz, \end{aligned}$$

where the operators D_1, D_2 and ∇_1 are

$$D_1 = y'^2 \frac{\partial^2}{\partial x'^2} - 2x'y' \frac{\partial^2}{\partial x' \, \partial y'} + x'^2 \frac{\partial^2}{\partial y'^2} - x' \frac{\partial}{\partial x'} - y' \frac{\partial}{\partial y'} - 1,$$

$$D_2 = y' \frac{\partial}{\partial x'} - x' \frac{\partial}{\partial y'}, \qquad \nabla_1^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} \cdot$$

To satisfy the first equation in (1), we have

$$\begin{aligned} (\lambda + \mu) \frac{\partial \Delta}{\partial x'} + \mu\{\nabla_1^2 u_1 + k^2 D_1(u_1) - 2k^2 D_2(v_1)\} = 0, \\ (\lambda + \mu) \frac{\partial \Delta}{\partial y'} + \mu\{\nabla_1^2 v_1 + k^2 D_1(v_1) + 2k^2 D_2(u_1)\} = 0. \end{aligned} \tag{5}$$

The displacements which satisfy the above conditions, also satisfy the second equation in (1). From the third equation in (1) we have the relation

$$-k(\lambda + \mu) D_2(\Delta) + \mu\{\nabla_1^2 w_1 + k^2 D_1(w_1) + k^2 w_1\} = 0. \tag{6}$$

Accordingly, Eqs. (5) and (6) are the equations of equilibrium for this case. Integrating these differential equations, we obtain u_1, v_1 and w_1, which are independent of z. Substituting them in the following equations

$$\begin{aligned}
X'_{x'} &= \lambda\Delta + 2\mu \frac{\partial u_1}{\partial x'}, \\
Y'_{y'} &= \lambda\Delta + 2\mu \frac{\partial v_1}{\partial y'}, \\
Z_z &= \lambda\Delta + 2\mu(\beta - kD_2(w_1)), \\
X'_{y'} &= \mu\left(\frac{\partial u_1}{\partial y'} + \frac{\partial v_1}{\partial x'}\right), \\
X'_z &= \mu\left(\frac{\partial w_1}{\partial x'} - kD_2(u_1) + kv_1 - \alpha y'\right), \\
Y'_z &= \mu\left(\frac{\partial w_1}{\partial y'} - kD_2(v_1) - ku_1 + \alpha x'\right),
\end{aligned} \tag{7}$$

we obtain the stresses, which are independent of z.

If we express the equation representing the bounding curve of the section by

$$F(x', y') = 0, \tag{8}$$

the condition that the bounding surface of the rod is free from traction is satisfied if the equations

$$\begin{aligned}
X'_{x'} \frac{\partial F}{\partial x'} + X'_{y'} \frac{\partial F}{\partial y'} - kD_2(F)X'_z &= 0, \\
X'_{y'} \frac{\partial F}{\partial x'} + Y'_{y'} \frac{\partial F}{\partial y'} - kD_2(F)Y'_z &= 0, \\
X'_z \frac{\partial F}{\partial x'} + Y'_z \frac{\partial F}{\partial y'} - kD_2(F)Z_z &= 0,
\end{aligned} \tag{9}$$

hold at all points of the bounding curve of the section.

Let us consider the equilibrium of a portion of the rod cut by two parallel planes perpendicular to the axis of the helix. Since the stresses in Eq. (7) are independent of z, the resultant of the shearing stress on each plane is the same in magnitude but generally different in direction; and from the condition of equilibrium of tractions, the resultant of the shearing stress vanishes. The effect of the normal traction on each plane is statically equivalent to a single force and a couple. The former can be cancelled by taking the constant β so as to satisfy the condition

$$\iint Z_z \, dx' \, dy' = 0, \tag{10}$$

and by a similar consideration, we see that the latter vanishes. Accordingly, the effect

of all tractions acting on each plane is equivalent to a couple due to the shearing force, and the solution which satisfies Eqs. (5) and (6) with the boundary conditions (9) and (10) is the one for the torsion problem of the rod. If we determine α so as to satisfy the condition that the moment of the couple due to the shearing force vanishes on each plane, instead of employing condition (10), then we have a solution for the problem of the stretching of a rod.

First, we shall consider the torsion problem. When k is small, u_1, v_1 and Δ are small quantities since they all vanish for a straight rod, and so if we neglect the smaller quantities of the second order, the equations of equilibrium (5) and (6) can be written in simpler forms as

$$\begin{aligned} \frac{\partial \Delta}{\partial x'} + \frac{\mu}{\lambda + \mu} \nabla_1^2 u_1 &= 0, \\ \frac{\partial \Delta}{\partial y'} + \frac{\mu}{\lambda + \mu} \nabla_1^2 v_1 &= 0, \\ \nabla_1^2 w_1 &= 0, \end{aligned} \tag{11}$$

and the third equation of the boundary condition (9) becomes

$$X_z' \frac{\partial F}{\partial x'} + Y_z' \frac{\partial F}{\partial y'} = 0, \tag{12}$$

and the stresses X_z' and Y_z' become

$$X_z' = \mu\left(\frac{\partial w_1}{\partial x'} - \alpha y'\right), \qquad Y_z' = \mu\left(\frac{\partial w_1}{\partial y'} + \alpha x'\right). \tag{13}$$

Accordingly, when k is small, the shearing stresses X_z' and Y_z' and hence the torque acting on the rod are the same as those for a straight cylinder, but the other stresses do not vanish as in the latter case.

Let us put

$$w_1 = i(f_3 - \bar{f}_3), \tag{14}$$

where f_3 is an arbitrary function and $f_3 = f_3(\zeta)$, $\bar{f}_3 = f_3(\bar{\zeta})$, $\zeta = x' + iy'$ and $\bar{\zeta} = x' - iy'$. w_1 satisfies the third equation in (11). Substituting this expression for w_1 into the first and second equations in (11), we have

$$\begin{aligned} \frac{\lambda + 2\mu}{\lambda + \mu}\frac{\partial^2 u_1}{\partial x'^2} + \frac{\partial^2 v_1}{\partial x'\,\partial y'} + \frac{\mu}{\lambda + \mu}\frac{\partial^2 u_1}{\partial y'^2} &= k[f_3' + \bar{f}_3' + \zeta f_3'' + \bar{\zeta}\bar{f}_3''], \\ \frac{\mu}{\lambda + \mu}\frac{\partial^2 v_1}{\partial x'^2} + \frac{\partial^2 u_1}{\partial x'\,\partial y'} + \frac{\lambda + 2\mu}{\lambda + \mu}\frac{\partial^2 v_1}{\partial y'^2} &= ik[f_3' - \bar{f}_3' + \zeta f_3'' - \bar{\zeta}\bar{f}_3'']. \end{aligned} \tag{15}$$

The particular solution of Eq. (15) is

$$u_1 = k \int f_3'\zeta\, d\zeta + k \int \bar{f}_3'\bar{\zeta}\, d\bar{\zeta}, \qquad v_1 = 0.$$

Setting the right sides of Eqs. (15) equal to zero, we obtain the relations

$$\nabla_1^4 u_1 = 0, \qquad \text{or} \qquad \nabla_1^4 v_1 = 0, \tag{16}$$

from which we can put

$$u_1 = f_1 + \bar{f}_1 + x'(f_2 + \bar{f}_2),$$

where f_1 and f_2 are arbitrary functions of ζ. Inserting this expression for u_1 into (15), we have

$$v_1 = i(f_1 - \bar{f}_1) + ix'(f_2 - \bar{f}_2) + i\frac{\lambda + 3\mu}{\lambda + \mu}\left[\int f_2\, d\zeta - \int \bar{f}_2\, d\bar{\zeta}\right].$$

Accordingly, the displacements which satisfy the equations of equilibrium (11) can be expressed in the forms

$$\begin{aligned} u' &= f_1 + \bar{f}_1 + x'(f_2 + \bar{f}_2) + k\left[\int f_3'\zeta\, d\zeta + \int \overline{f_3'\zeta}\, d\bar{\zeta}\right] - \alpha y'z, \\ v' &= i(f_1 - \bar{f}_1) + ix'(f_2 - \bar{f}_2) + i(2p + 1)\left[\int f_2\, d\zeta - \int \bar{f}_2\, d\bar{\zeta}\right] + \alpha x'z, \\ w &= i(f_3 - \bar{f}_3) + \beta z, \end{aligned} \tag{17}$$

and the corresponding expressions for the stresses are given in the forms

$$\begin{aligned} X'_{x'} &= 2\mu\{f_1' + \bar{f}_1' + p(f_2 + \bar{f}_2) + x'(f_2' + \bar{f}_2') + k(\zeta f_3' + \bar{\zeta}\bar{f}_3')\} + \lambda\beta, \\ Y'_{y'} &= -2\mu\{f_1' + \bar{f}_1' + (p + 2)(f_2 + \bar{f}_2) + x'(f_2' + \bar{f}_2')\} + \lambda\beta, \\ Z_z &= -2\mu\{(1 - p)(f_2 + \bar{f}_2) + k(\zeta f_3' + \bar{\zeta}\bar{f}_3')\} + (\lambda + 2\mu)\beta, \\ X'_{y'} &= 2i\mu\left\{f_1' - \bar{f}_1' + x'(f_2' - \bar{f}_2') + (p + 1)(f_2 - \bar{f}_2) + \frac{k}{2}(\zeta f_3' - \bar{\zeta}\bar{f}_3')\right\}, \\ X'_z &= \mu\{i(f_3' - \bar{f}_3') - \alpha y'\}, \\ Y'_z &= -\mu\{f_3' + \bar{f}_3' - \alpha x'\}, \end{aligned} \tag{18}$$

where

$$p = \frac{\mu}{\lambda + \mu}.$$

As an example, we shall consider an elliptic spiral rod whose section is given by

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1. \tag{19}$$

Let us put

$$2f_1' = A_0 + A_1\zeta^2, \qquad 2f_2 = B_0 + B_1\zeta^2, \qquad 2f_3' = h\zeta, \tag{20}$$

where

$$h = \frac{a^2 - b^2}{a^2 + b^2}\alpha.$$

Substituting the expressions in (20) into (18), we obtain

$$
\begin{aligned}
X'_{x'} &= 2\mu\{A_0 + pB_0 + 2B_1x'^2 + (A_1 + pB_1 + kh)(x'^2 - y'^2)\} + \lambda\beta, \\
Y'_{y'} &= -2\mu\{A_0 + (p + 2)B_0 + 2B_1x'^2 + [A_1 + (p + 2)B_1](x'^2 - y'^2)\} + \lambda\beta, \\
Z_z &= -2\mu\{(1 - p)B_0 + [(1 - p)B_1 + kh](x'^2 - y'^2)\} + (\lambda + 2\mu)\beta, \\
X'_{y'} &= -2\mu\{2A_1 + 2(p + 2)B_1 + kh\}x'y', \\
X'_z &= -\mu(h + \alpha)y', \\
Y'_z &= -\mu(h - \alpha)x'.
\end{aligned} \tag{21}
$$

From the boundary conditions (9) and (10) we obtain a system of simultaneous equations for the determination of the unknown constants A_0, B_0, A_1, B_1 and β as

$$
\begin{aligned}
&\frac{A_0}{a^2} + \frac{p}{a^2}B_0 + A_1 + (p + 2)B_1 + \frac{\lambda\beta}{2\mu a^2} + kh = 0, \\
&\frac{A_0}{b^2} + \frac{p + 2}{b^2}B_0 - A_1 - (p + 2)B_1 - \frac{\lambda\beta}{2\mu b^2} = 0, \\
&\left(\frac{1}{a^2} + \frac{3}{b^2}\right)A_1 + \left(\frac{p}{a^2} + \frac{3p + 6}{b^2}\right)B_1 + \left(\frac{1}{a^2} + \frac{3}{b^2}\right)kh = 0, \\
&\left(\frac{3}{a^2} + \frac{1}{b^2}\right)A_1 + \left(\frac{3p + 6}{a^2} + \frac{p + 4}{b^2}\right)B_1 = 0, \\
&(1 - p)B_0 + \frac{(a^2 - b^2)}{4}[(1 - p)B_1 + kh] - \frac{\lambda + 2\mu}{2\mu}\beta = 0.
\end{aligned} \tag{22}
$$

As a numerical example, we shall take the dimensions of the section as $a = 2$cm, $b = 1$ cm and shall assume the values of the elastic constants as $\lambda = 8.66 \times 10,^5$ $\mu = 8.20 \times 10^5$ (unit Kg weight per cm^2). Substituting these values into (22) and solving the simultaneous equations, we find (unit $k\alpha$)

$$
A_0 = -0.5786, \qquad A_1 = -1.6793, \qquad B_0 = 0.1156, \qquad B_1 = 0.4627,
$$
$$
\beta = 0.4500.
$$

In our calculation, the constant β does not vanish. Hence we see that when a spiral rod is twisted, an axial elongation (or contraction) takes place. Substituting these numerical values of constants into Eq. (21), we calculate the stresses on the x' and y' axes, which are plotted in Figs. 1 and 2. In this case, the predominating stress which is concerned in torque is X'_z and its maximum amount is 1.6 $\mu\alpha$. While the predominating stress among the other stresses is the normal stress Z_z, which attains its maximum amount of 5.44 $k\mu\alpha$ at the points $x' = \pm 2$, $y' = 0$, as is shown in the figures. Accordingly,

the latter can not be ignored compared with the former except in cases when k is extremely small.

The shape of the cross section is not deformed by twist in a straight rod, but in the

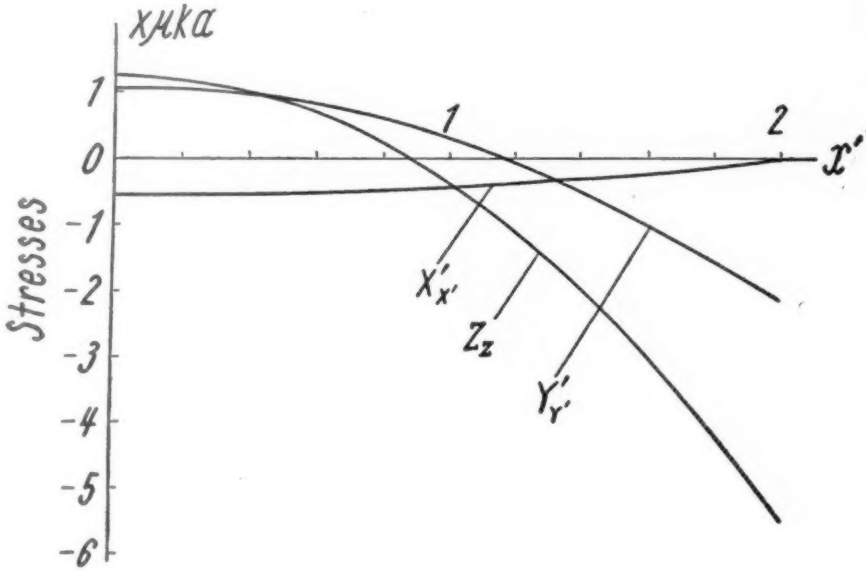

FIG. 1. Stresses on x' axis.

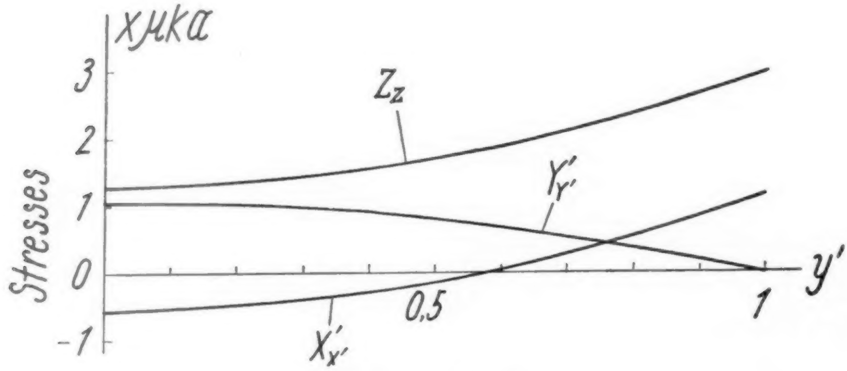

FIG. 2. Stresses on y' axis.

case of a spiral rod the distortion of the section, as shown in Fig. 3, is caused by twist.

Next, we shall treat the stretching problem. For convenience of calculations, let us put the displacements in the forms

$$u' = u_1(x', y') - \gamma x' - \alpha y'z,$$
$$v' = v_1(x', y') - \gamma y' + \alpha x'z, \tag{23}$$
$$w = w_1(x', y') + \beta z,$$

where

$$\gamma = \frac{1}{2}(1 - p)\beta.$$

When k is small, neglecting the smaller quantities of the order of k^3, the further calculations become quite similar to those of the previous case, viz., Eq. (11) is equally

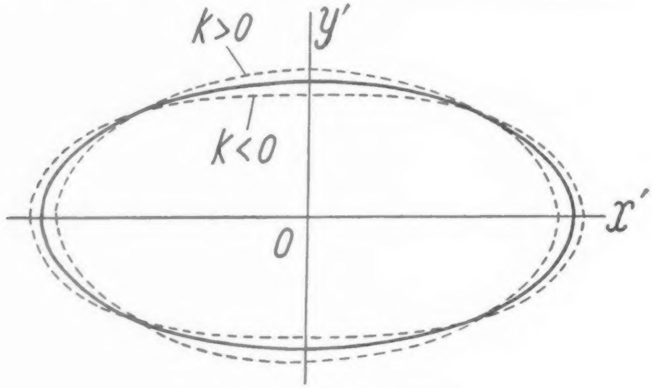

FIG. 3. Distortion of the section.

valid for this case and the stresses are represented in the same forms as in Eq. (18), except the following stresses

$$X'_{z'} = 2\mu\{f_1' + \overline{f_1'} + p(f_2 + \overline{f_2}) + x'(f_2' + \overline{f_2'}) + k(\zeta f_3' + \overline{\zeta f_3'})\},$$
$$Y'_{y'} = -2\mu\{f_1' + \overline{f_1'} + (p + 2)(f_2 + \overline{f_2}) + x'(f_2' + \overline{f_2'})\}, \tag{24}$$
$$Z_z = -2\mu\left\{(1 - p)(f_2 + \overline{f_2}) + k(\zeta f_3' + \overline{\zeta f_3'}) - \frac{1}{2}(3 - p)\beta\right\}.$$

For an elliptic section, we shall put

$$2f_1' = A_0 + A_1\zeta^2, \qquad 2f_2 = B_0 + B_1\zeta^2, \qquad 2f_3' = C_0\zeta. \tag{25}$$

The corresponding stresses become

$$X'_{z'} = 2\mu\{A_0 + pB_0 + 2B_1x'^2 + (A_1 + pB_1 + kC_0)(x'^2 - y'^2)\},$$
$$Y'_{y'} = -2\mu\{A_0 + (p + 2)B_0 + 2B_1x'^2 + [A_1 + (p + 2)B_1](x'^2 - y'^2)\},$$
$$Z_z = -2\mu\left\{(1 - p)B_0 + [(1 - p)B_1 + kC_0](x'^2 - y'^2) - \frac{1}{2}(3 - p)\beta\right\}, \tag{26}$$

$$X'_{y'} = -2\mu\{2A_1 + 2(p+2)B_1 + kC_0\}x'y',$$

$$X'_z = -\mu(C_0 + \alpha)y',$$

$$Y'_z = -\mu(C_0 - \alpha)x'.$$

From the boundary condition (9) with the condition that the moment of couple due to the shearing stresses vanishes, we have a system of simultaneous equations for the determination of the unknown constants A_0, A_1, B_0, B_1, C_0 and α, as

$$(a^2 - b^2)C_0 = (a^2 + b^2)\alpha,$$

$$(a^2 + b^2)C_0 = (a^2 - b^2)\{\alpha + (3-p)k\beta\},$$

$$\frac{A_0}{a^2} + \frac{p}{a^2}B_0 + A_1 + (p+2)B_1 + kC_0 = 0,$$

$$\left(\frac{1}{a^2} + \frac{3}{b^2}\right)A_1 + \left(\frac{p}{a^2} + \frac{3p+6}{b^2}\right)B_1 + \frac{kC_0}{2}\left(\frac{1}{a^2} + \frac{5}{b^2}\right) + \frac{k\alpha}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = 0, \tag{27}$$

$$\frac{A_0}{b^2} + \frac{p+2}{b^2}B_0 - A_1 - (p+2)B_1 = 0,$$

$$\left(\frac{3}{a^2} + \frac{1}{b^2}\right)A_1 + \left(\frac{3p+6}{a^2} + \frac{p+4}{b^2}\right)B_1 + \frac{kC_0}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - \frac{k\alpha}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = 0.$$

If q is the mean value of the axial normal stress over the section, then

$$\begin{aligned} q &= \frac{1}{\pi ab}\iint Z_z\, dx'\, dy' \\ &= \frac{E}{3-p}\left\{(3-p)\beta - 2(1-p)B_0 - \frac{1}{2}(a^2 - b^2)[(1-p)B_1 + kC_0]\right\}, \end{aligned} \tag{28}$$

where E is Young's modulus. From Eq. (28), β can be expressed as a multiple of q.

We shall consider an elliptic section of the same dimensions as in the previous example. From Eqs. (27) and (28) we have

$$C_0 = 2.3565k\beta, \qquad \alpha = 1.4139k\beta, \qquad A_0 = -1.8851k^2\beta,$$

$$B_0 = 0, \qquad A_1 = -4.8148k^2\beta, \qquad B_1 = 1.1783k^2\beta,$$

and

$$\beta = \frac{q}{E\{1 - 1.7673k^2\}}\,.$$

Accordingly a twist arises when a spiral rod is pulled axially, since α does not vanish in our calculations. The normal stresses $X'_{x'}$, $Y'_{y'}$ attain their maximum value of 1.5 $k^2E\beta$ at the center of the section, and so they are very small quantities compared with Z_z. The normal stress Z_z is distributed almost uniformly over the section and it attains

its maximum and minimum values of $(1 + 2.356k^2)E\beta$ and $(1 - 9.425\ k^2)E\beta$ at both ends of the minor and major axes of the ellipse respectively. The predominating stress among the shearing stresses is X'_z and it attains its maximum value of $1.5\ kE\beta$ at both ends of the minor axis.

In conclusion, the writer wishes to express his thanks to Miss E. Itagaki, his assistant, for her earnest help in this study.

THE EFFECT OF INITIAL DEFORMATIONS ON THE BEHAVIOUR OF A CYLINDRICAL SHELL UNDER AXIAL COMPRESSION*

BY

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The theoretical investigation presented here concerns the effect of certain initial deformations on the buckling under axial compression of thin plate structures. A general theory is developed and used to make approximate analyses of the complete cylinder and curved panel. Two extreme cases of the curved panel are treated: (a) uniform longitudinal shortening and, (b) uniform longitudinal stress. Particular attention is paid to the possibility of inducing buckling in a favorable mode, and thus increasing the load which a structure can withstand.

Introduction

The buckling of a cylindrical panel under axial compression has recently been the subject of a number of investigations using the non-linear theory of thin plates. This theory is valid for deflections of the same order of magnitude as the thickness of the panel.

The non-linear theory has shown that, for a panel with perfect initial form and starting from the configuration of infinitesimal deflections which correspond to the critical loads of the linear theory, buckling may evolve through one of several paths. While some of these paths are favorable and give loads which increase with increasing deformations, others cause the load to fall after buckling. This has been proved quantitatively by von Kármán and Tsien[1] (1) and, independently, also by the author (2).

Experiments have shown that the effect of initial deformations is usually to cause buckling in an unfavorable mode. However, Welter (3) has shown from compression tests on curved panels, that it is possible to delay buckling by initially bending a panel to a smaller radius than that at which it is tested. These results suggested the present investigation.

The General Equations for a Cylindrical Shell with Initial Deformations

Let t' be the thickness of the shell and write $t = t'/\sqrt{12(1-\nu^2)}$ where ν is Poisson's ratio. We call state "0" a state of deformation by which the mean surface takes the form of a circular cylinder whose radius is R; let x and y be the axial and circumferential coordinates, w the radial displacements (positive outwards), leading from state "0" to the actual state "A".

Assuming that t' is small compared with R and that the displacements w have the same order of magnitude as t', the normal and shearing stresses in state "A" may be represented in the forms

$$\sigma_x = E\frac{\partial^2 f}{\partial y^2}, \qquad \sigma_y = E\frac{\partial^2 f}{\partial x^2}, \qquad \tau = -E\frac{\partial^2 f}{\partial x\,\partial y}, \tag{1}$$

*Received August 28, 1950.

[1]Numbers in parentheses refer to Bibliography at the end of the paper.

where E is Young's modulus and f is a stress function. Similarly the stresses in state "0" can be derived from another stress function f'. If the material is completely elastic, the stress variation from state "0" to state "A" is related to stretching by the equation

$$\nabla^4(f - f') + k = 0, \tag{2}$$

where

$$k = \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\right) - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2.$$

The shell is held in state "0" by applying a certain distribution of normal pressure $Et'p_0$, where p_0 is, in general, a funnction of x and y; the quantity

$$p' = p_0 - \frac{1}{R}\frac{\partial^2 f'}{\partial x^2} \tag{3}$$

will be zero if the shell is free from transverse shearing stresses in state "0". In general, this will not be the case; the quantity $Et'p'$ represents a *net inherent pressure* that adds to the *plate* and *membrane* pressures in any state of the shell. Therefore, the equilibrium condition in state "A", under the applied pressure $Et'p$ yields

$$p = p' + t^2\nabla^4 w - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\right)\frac{\partial^2 f}{\partial x^2} + 2\,\frac{\partial^2 w}{\partial x\,\partial y}\,\frac{\partial^2 f}{\partial x\,\partial y}\,. \tag{4}$$

If the shell is free from stress in state "0", the functions f' and p' are both zero and Eqs. (2) and (4) degenerate to the known equations for a shell having perfect initial form and no initial stresses.

If a particular state $w = w^*$ exists for which the shell is free from stress when there is no normal pressure, we may write

$$f' = k^*, \qquad p' = -t^2\nabla^4 w^*$$

where k^* is the value for k when $w = w^*$. Eqs (2) and (4) then take the respective forms

$$\nabla^4 f + k - k^* = 0 \tag{5}$$

$$p = t^2\nabla^4(w - w^*) - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\right)\frac{\partial^2 f}{\partial x^2} + 2\,\frac{\partial^2 w}{\partial x\,\partial y}\,\frac{\partial^2 f}{\partial x\,\partial y}\,.$$

These equations have already been presented by the writer (2). They have been used by Dei Poli (4) in his studies on the cylindrical panel with initial deformations.

The Complete Cylindrical Shell with Initial Deformations

We consider the case of a cylindrical shell with the particular deformations represented by

$$\begin{aligned} f' &= 0 \\ p' &= p_1 \sin\xi \sin\varphi + p_2 \cos 2\xi + p_3 \cos 2\varphi + p_4 \sin 3\xi \sin\varphi \end{aligned} \tag{6}$$

with $\xi = \pi x/a$, $\varphi = \pi y/b$, where a and b are constants and $\pi R/b$ is any integer. The first of these equations states that the shell is free from stress in state "0". The con-

strained sections of the shell are assumed to be so remotely situated that their influence on the buckling process may be neglected.

To obtain an approximate solution to the problem of the shell under axial load, we first express the functions f and w in the following non-dimensional forms

$$\begin{aligned} f/t^2 &= f_1 \sin \xi \sin \varphi + f_2 \cos 2\xi + f_3 \cos 2\varphi + f_4 \sin 3\xi \sin \varphi - \eta y^2/Rt, \\ w/t &= w_1 \sin \xi \sin \varphi + w_2 \cos 2\xi + w_3 \cos 2\varphi + w_4 \sin 3\xi \sin \varphi + w_0 . \end{aligned} \tag{7}$$

The coefficient η, which is positive for compressive axial loads, represents the ratio of the mean axial stress to the critical stress $\sigma_c = -2Et/R$ derived from the linear theory. The last term in the first of Eqs. (7) therefore relates to the mean axial stress.

To determine the values of the constants f_1, f_2, f_3, f_4, w_1, w_2, w_3 and w_4, we use the Galerkin method and evaluate the integral expressions

$$\int_0^{2a} dx \int_0^{2b} [\nabla^4 f + k]\psi_1 \, dy = \int_0^{2a} dx \int_0^{2b} p\psi_2 \, dy = 0 \tag{8}$$

where for ψ_1 and ψ_2 we replace successively the functions

$$\sin \xi \sin \varphi; \qquad \cos 2\xi; \qquad \cos 2\varphi; \qquad \sin 3\xi \sin \varphi$$

In this way we obtain

$$D_1 f_1 = -\beta^2 w_1 + 2w_1 w_2 + 2w_1 w_3 - 2w_2 w_4 , \tag{9}$$

$$32\beta^2 f_2/\alpha^2 = -8\beta^2 w_2 + w_1^2 - 2w_1 w_4 , \tag{10}$$

$$32\alpha^2 f_3/\beta^2 = w_1^2 + 9w_4^2 , \tag{11}$$

$$D_4 f_4 = -9\beta^2 w_4 - 2w_1 w_2 + 18 w_3 w_4 , \tag{12}$$

$$D_1 w_1 = 2\eta\beta^2 w_1 + \beta^2 f_1 - 2w_1 f_2 - 2w_2 f_1 - 2w_1 f_3 - 2w_3 f_1 + 2w_2 f_4 + 2w_4 f_2 + c_1 , \tag{13}$$

$$16\beta^2 w_2/\alpha^2 = 8\eta\beta^2 w_2 + 4\beta^2 f_2 - w_1 f_1 + w_1 f_4 + w_4 f_1 + c_2 , \tag{14}$$

$$16\alpha^2 w_3/\beta^2 = -w_1 f_1 - 9w_4 f_4 + c_3 , \tag{15}$$

$$D_4 w_4 = 18\eta\beta^2 w_4 + 9\beta^2 f_4 + 2w_1 f_2 + 2w_2 f_1 - 18 w_3 f_4 - 18 w_4 f_3 + c_4 , \tag{16}$$

where

$$D_1 = \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)^2, \qquad D_4 = \left(\frac{\alpha}{3\beta} + \frac{3\beta}{\alpha}\right)^2,$$

$$\alpha = a/\pi\sqrt{Rt}, \qquad \beta = b/\pi\sqrt{Rt},$$

$$c_i = a^2 b^2 p_i/\pi^4 t^3, \qquad (i = 1, 2, 3, 4).$$

Taking into account the equation

$$\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{\partial^2 f}{\partial y^2} - \nu \frac{\partial^2 f}{\partial x^2} \tag{17}$$

where u is the axial component of displacement, the mean axial negative shortening

$$\epsilon_x = \frac{1}{2a} \int_0^{2a} \frac{\partial u}{\partial x} dx \tag{18}$$

may be calculated. The condition of uniform shortening, requiring that ϵ_x be independent of φ, is satisfied by Eqs. (7), when Eq. (11) is considered. Denoting by ϵ the constant value of ϵ_x, we obtain

$$-\epsilon R/t = 2\eta + (w_1^2 + 8w_2^2 + 9w_4^2)/8\alpha^2. \tag{19}$$

For a given initial deformation and given wave length, the constants c_i are first calculated from Eqs. (9)-(16) by replacing for the w's the values w_1', w_2', w_3', w_4' that

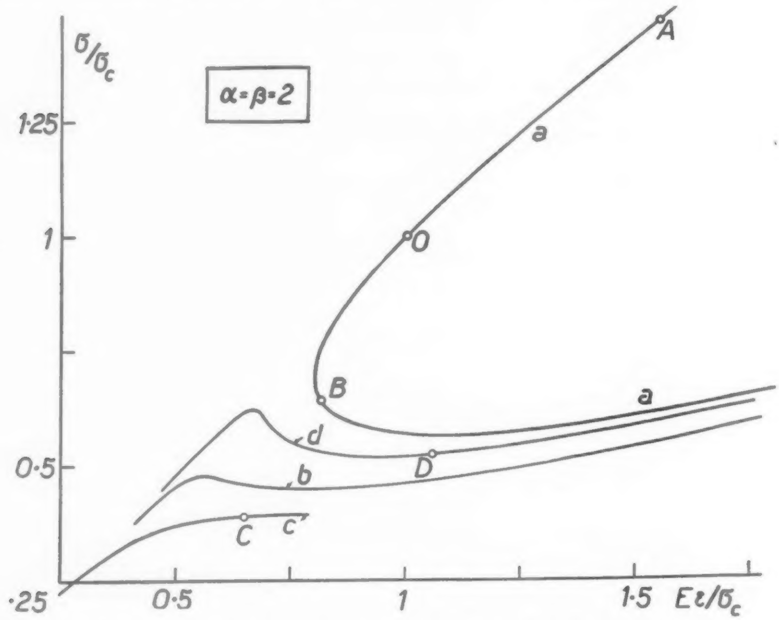

FIG. 1. Stress-strain diagram for the complete cylinder.

Line a ($w_1' = w_2' = 0$). Line b ($w_1' = w_2' = 1/4$). Line c ($w_1' = 2$, $w_2' = -1/4$). Line d ($w_1' = 1/2$, $w_2' = -1/16$).

correspond to $\eta = 0$. Then successive sets of values of $\eta, f_1, f_2, f_3, f_4, w_1, w_2, w_3, w_4$ satisfying Eqs. (9)-(16) are to be found. The corresponding stress, waveform and shortening can be deduced from Eqs. (7) and (19).

(a) First form of deformation equation

As a first step, we consider the case where

$$\alpha = \beta, \qquad w_3 = w_4 = f_4 = 0$$

and hence disregard Eqs. (12), (15) and (16). In this case, the results obtained for a

special value of β may be generalized by considering that the parameter β disappears from the equations if the quantities

$$f_1/\beta^4,\ f_2/\beta^4,\ f_3/\beta^4,\ w_1/\beta^2,\ w_2/\beta^2,\ c_1/\beta^6,\ c_2/\beta^6,\ (\eta/\beta^2) - (2/\beta^4)$$

are introduced.

The results of the calculations for $\beta = 2$ are represented in Figs. 1 and 2. In Fig. 1, values of $\eta = \sigma/\sigma_c$ are plotted against $E\epsilon/\sigma_c$ for various values of w_1' and w_2'. Considering

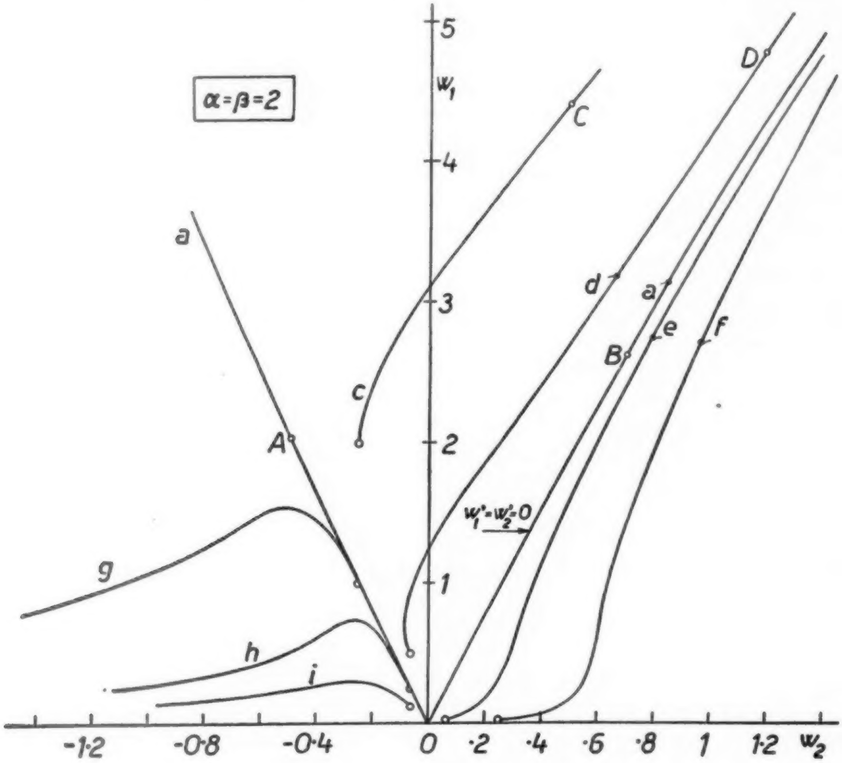

FIG. 2. Displacement components w_1 versus w_2.

Line a ($w_1' = w_2' = 0$). Line c ($w_1' = 2$, $w_2' = -1/4$). Line d ($w_1' = 1/2$, $w_2' = -1/16$).
Line e ($w_1' = 1/32$, $w_2' = 1/16$). Line f ($w_1' = 1/32$, $w_2' = 1/4$).
Line g ($w_1' = 1$, $w_2' = -1/4$). Line h ($w_1' = 1/4$, $w_2' = -1/16$).
Line i ($w_1' = 1/8$, $w_2' = -1/16$).

the graph for a shell of perfect initial form ($w_1' = w_2' = 0$), O is the point at which buckling begins: for this point, $w_1 = w_2 = 0$ and $\eta = 1$. The lower branch of the curve through B represents the unfavorable path with decreasing stresses and strains accompanying the first phase of increasing deformations. The upper branch of the curve through A represents the favorable path. If Hooke's law were obeyed and the curve followed the upper branch, the behaviour of the shell would be practically unaffected by buckling.

Figure 2 shows graphs of the deflection components w_1 versus w_2 for various initial values w_1' and w_2'. These graphs are limited to positive values of w_1, since the diagrams would be symmetrical with respect to the w_2 axis. The graph for a shell with $w_1' = w_2' = 0$, indicates that, for small values of w_1 and w_2, $w_1 \approx \pm 4w_2$; the positive sign corresponds to the unfavorable branch of the curve in Fig. 1. A third possible state of equilibrium would be represented in Fig. 1 by a horizontal line through 0; this corresponds to axially symmetrical buckling with $w_1 = 0$.

From Fig. 2, the evolution of the buckled form of the cylinder may be traced for various initial deformations, the corresponding stress-strain diagrams being given in Figs. 1, 3 and 5[2]. The graphs of w_1 vs. w_2 in Fig. 2 are of two kinds: (a) those which approach that branch of the line for a cylinder of perfect initial form which has positive slope and, (b) those which tend to approach the $-w_2$ axis; in any case they diverge from line 0A. From Figs. 1, 3 and 5, it may be seen that the first kind of initial deformation leads to an unfavorable path while the second kind indicates a rather favorable behaviour with the load approaching the critical load of the linear theory. It will be seen later that this latter possibility is fictitious.

(b) Second type of deflection equation

Calculations taking into account the w_3 component of displacement and neglecting w_4 and f_4 as before, have already been carried out (2) for the case of a cylindrical shell with perfect initial form—that is, with $c_1 = c_2 = c_3 = 0$. The effect of the w_3 component does not modify the results to any great extent. For example, the dotted line k in Fig. 5 shows the stress-strain curve for $\beta = 2$ and $w_1' = 1/32$, $w_2' = 1/16$, $c_3 = 0$. This curve does not differ substantially from the full line e which corresponds to the same initial deformation, but with w_3 and Eq. (15) neglected.

(c) Third type of deflection equation

The stress and displacement functions will now be expressed in the forms

$$\begin{aligned} f/t^2 &= f_1 \sin\xi \sin\varphi + f_{II} \cos\xi \sin\varphi + f_2 \cos 2\xi + f_3 \cos 2\varphi - \eta y^2/Rt, \\ w/t &= w_1 \sin\xi \sin\varphi + w_{II} \cos\xi \sin\varphi + w_2 \cos 2\xi + w_0 \,. \end{aligned} \tag{20}$$

The coefficients are determined from Eqs. (8) in which we write successively

$$\psi_1 = \psi_2 = \sin\xi \sin\varphi, \; = \cos\xi \sin\varphi, \; = \cos 2\xi, \qquad \psi_1 = \cos 2\varphi.$$

By eliminating the coefficients f, we obtain for $\alpha = \beta$:

$$\begin{aligned} 2\eta\beta^2 - 4 - \frac{\beta^4}{4} &= -\frac{3\beta^2}{2} w_2 + \frac{w_1^2}{8} + w_2^2 + \frac{c_1}{w_1} = \frac{3\beta^2}{2} w_2 + \frac{w_{II}^2}{8} + w_2^2 + \frac{c_{II}}{w_{II}} \\ &= -\frac{3\beta^2}{32 w_2}(w_1^2 - w_{II}^2) + \frac{1}{8}(w_1^2 + w_{II}^2) + \frac{c_2}{w_2}\,. \end{aligned} \tag{21}$$

The constants c_1, c_{II} and c_2 depend upon the values w_1', w_{II}' and w_2' of the displacement components when $\eta = 0$.

[2]Corresponding curves are marked by the same letter.

It should be noted that the solution does not modify if the quantities w_{I} , w_{II} , w_2 are respectively replaced by w_{II} , w_{I} , $-w_2$.

As before, calculations were made for $\beta = 2$. For the initial displacement components ($w_{\mathrm{I}}' = 1$, $w_{\mathrm{II}}' = 1/32$, $w_2' = -1/4$), the stress-strain diagram is represented by the full line in Fig. 3. Comparing this graph with line g, corresponding to

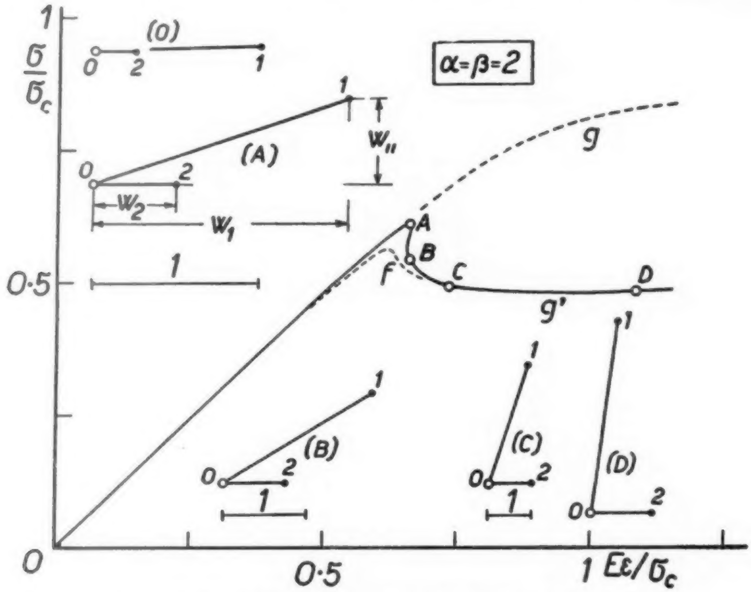

Fig. 3. Stress-strain diagram for the complete cylinder.

Line f ($w_{\mathrm{I}}' = 1/32$, $w_{\mathrm{II}}' = 0$, $w_2' = 1/4$), or ($w_{\mathrm{I}}' = 0$, $w_{\mathrm{II}}' = 1/32$, $w_2' = -1/4$).
Line g ($w_{\mathrm{I}}' = 1$, $w_{\mathrm{II}}' = 0$, $w_2' = -1/4$).
Line g' and vector diagrams (0), (A), (B), (C), (D) of displacement components: ($w_{\mathrm{I}}' = 1$, $w_{\mathrm{II}}' = 1/32$, $w_2' = -1/4$).

($w_{\mathrm{I}}' = 1$, $w_2' = -1/4$), the effect of the additional displacement term is seen to change the form from a favorable to an unfavorable one.

The evolution of the buckled form will be evident from the vector diagrams (0), (A), (B), (C) and (D) in Fig. 3, which represent the displacements for the corresponding points on the full curve. In these diagrams, the vector "02" represents the component w_2 , the horizontal and vertical components of the vector "01" represent respectively w_{I} and w_{II} . Diagram (0), corresponding to $\eta = 0$, indicates that w' is negligible. However, as one proceeds from (0) to (D), w_{II} increases rapidly, while w_{I} decreases. The units for displacements are indicated on the diagrams.

Fig. 4 represents the deflections along a generator $y = 0$ or $y = nb$ (n is any positive integer), for points corresponding to those on the full curve in Fig. 3.[3] The initial form

[3]The position of the line $w = 0$ with respect to the curves in Fig. 4 is of no great interest: the value of w_0 in the expression for w is determined from the condition that the mean circumferential stress is zero.

The diagrams in Fig. 4 are drawn to such a scale that the quantity $\sqrt{w_{\mathrm{I}}^2 + w_{\mathrm{II}}^2}$ is always represented by the distance apart of the horizontal lines.

in Fig. 4—graph 0—, almost symmetrical about the verticals $x = -a/2, a/2, 3a/2$, etc., corresponds to the most favorable type. This symmetry is destroyed as the load increases and, during the buckling, the form approaches the unfavorable type which has a shape similar to graph (0), but with opposite signs for the deflections.

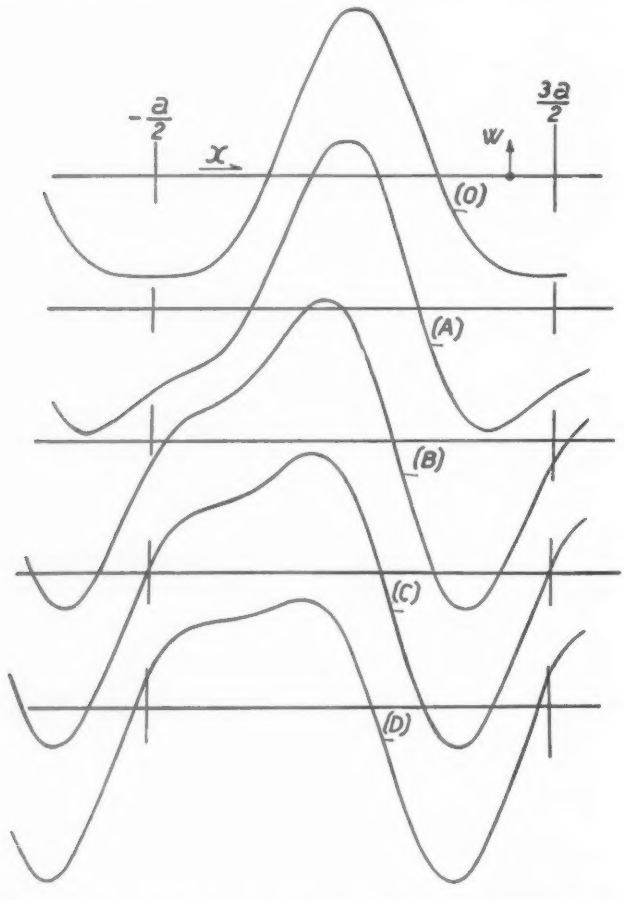

Fig. 4. Evolution of the deformation of a generator during the buckling process corresponding to line g′ in Fig. 3; w positive outward

Analogous results are plotted in Fig. 5. As before, the evolution of the three deformation components is represented vectorially. Starting with the initial diagram (0), for which ($w_1' = 1/8$, $w_{II}' = 1/32$, $w_2' = -1/16$), the vector diagrams (A), (B) and (C) show, to various scales, the deformation components, in successive phases of buckling, when $-w_2$ has the values 1/8, 1/4 and 1/2. The corresponding points in the stress-strain diagram fall approximately on curve e. Starting with the initial diagram (0′), for which ($w_1' = 1/4$, $w_{II}' = 1/32$, $w_2' = -1/16$), in successive phases when $-w_2$ has

the values 1/2 and 1, the conditions represented by diagrams (D) and (E) are attained. The corresponding points lie practically on the same curve *e*.

These results show that the behaviour of the cylinder, as calculated when the components w_I, w_{II}, w_2 are considered, always approaches the most unfavorable path between the one determined neglecting the w_I component and the one neglecting the w_{II} component.

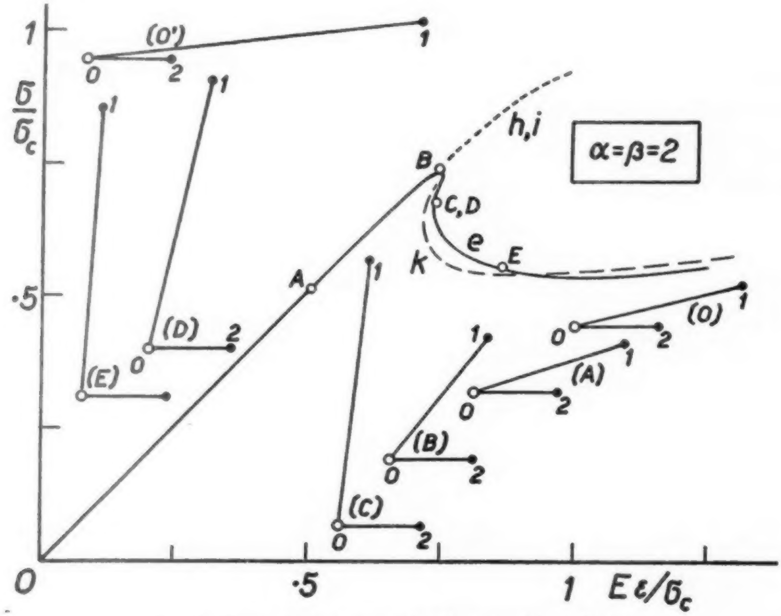

FIG. 5. Stress-strain diagram for the complete cylinder.

Line e ($w_I' = 1/32$, $w_{II}' = 0$, $w_2' = 1/16$), or ($w_I' = 0$, $w_{II}' = 1/32$, $w_2' = -1/16$).
Line h, i ($w_I' = 1/8 \div 1/4$, $w_{II}' = 0$, $w_2' = -1/16$).
Points A, B, C and vector diagrams (0), (A), (B), (C): ($w_I' = 1/8$, $w_{II}' = 1/32$, $w_2' = -1/16$).
Points D, E and vector diagrams (0'), (D), (E): ($w_I' = 1/4$, $w_{II}' = 1/32$, $w_2' = -1/16$).

The Curved Panel

The problem of a curved panel which is compressed in a direction parallel to the straight sides is now considered. It is assumed that the straight sides $y = 0$ and $y = b$ of the panel are simply supported and free from normal stresses and that the strain along these edges is constant (as if restrained by flexurally rigid, laterally and torsionally weak, axially very stiff side-stringers). The boundary conditions for $y = 0$ and $y = b$ are therefore

$$w = \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 f}{\partial x^2} = 0; \qquad \frac{\partial^2 f}{\partial y^2} = -\frac{2\eta t}{R}. \tag{22}$$

The parameter η represents here the ratio of the strain along the straight sides to the critical strain obtained from the linear theory for a complete cylinder of which the panel may be assumed to be a part.

For the curved edges $x = 0$ and $x = 2a$, we assume perfect clamping and hence

$$w = \frac{\partial w}{\partial x} = 0 \tag{23}$$

for these edges; in addition, we assume that these edges are free from shearing stress and write

$$\frac{\partial^2 f}{\partial x\, \partial y} = 0. \tag{24}$$

The problem is considered for two extreme cases: (a) uniform longitudinal shortening and, (b) uniform longitudinal stress.

(a) Panel subjected to uniform longitudinal shortening

For the stress and displacement functions, we write respectively

$$\begin{aligned} f/3\pi t^2 &= (f_1 \cos \xi + f_2 \cos 2\xi + f_3) \sin \varphi - \eta y^2/3\pi R t, \\ w/3\pi t &= [w_1(1 - \cos \xi) + w_2\,(1 - \cos 2\xi)] \sin \varphi, \end{aligned} \tag{25}$$

where $\xi = \pi x/a$, $\varphi = \pi y/b$. These functions satisfy all the boundary conditions given in Eqs. (22), (23) and (24). The condition of uniform axial shortening will be satisfied in an approximate form by writing[4]

$$\int_0^b (\epsilon_x - \epsilon) \sin \varphi \, dy = 0, \tag{26}$$

where $-\epsilon_x$ is the axial shortening calculated according to Eqs. (17) and (18) and $-\epsilon$ is the edge value given by $-\epsilon = 2\eta t/R$. Thus we get

$$\alpha^2 f_3/\beta^2 = -2w_1^2 - 8w_2^2\,. \tag{27}$$

Furthermore, to determine the constants f_1, f_2, f_3, w_1, w_2, we use the Galerkin method and evaluate the same Eqs. (8) in which, in this case, the field of integration is defined by the panel contour and for ψ_1, ψ_2 we write successively the functions

$$(1 - \cos \xi) \sin \varphi, \qquad (1 - \cos 2\xi) \sin \varphi.$$

Thus, assuming $f' = 0$, we obtain

$$\begin{aligned} D_1 f_1 &= \beta^2 w_1 + 8w_1^2 + 4w_1 w_2\,, \\ D_2 f_2 &= \beta^2 w_2 + 8w_2^2 + 8w_1 w_2 - \frac{3}{2} w_1^2\,, \\ 2\eta\beta^2 w_1 &= D_1 w_1 + 2\alpha^2(w_1 + w_2)/\beta^2 + (\beta^2 + 16w_1 - 4w_2) f_1 \\ &\quad + (32w_2 - 12w_1) f_2 - 8w_1 f_3 + c_1\,, \\ 2\eta\beta^2 w_2 &= D_2 w_2 + \alpha^2(w_1 + w_2)/2\beta^2 + (\beta^2 + 16w_2 + 8w_1) f_2 - w_1 f_1 - 8w_2 f_3 + c_2\,, \end{aligned} \tag{28}$$

[4]By substituting an arbitrary function of φ instead of $f_3 \sin \varphi$ in the expression for f and using $\epsilon_x = \epsilon$ to find the function, it is possible to satisfy exactly the condition that the shortening of all generators is constant.

where

$$D_1 = \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)^2, \qquad D_2 = \left(\frac{\alpha}{2\beta} + \frac{2\beta}{\alpha}\right)^2.$$

By elimination of f_1, f_2 and f_3, from Eqs. (27) and (28) we obtain two equations in w_1, w_2 and η, and hence a fourth degree equation in w_1 and w_2. This furnishes corresponding pairs of w_1 and w_2 and these may be used to determine the ratio η and the axial shortening.

The load sustained by the panel, expressed by

$$N = t' \int_0^b \sigma_x \, dy$$

varies with the section considered. We therefore define an equivalent mean stress σ as the stress which, when multiplied by the mean shortening and the volume of the panel,

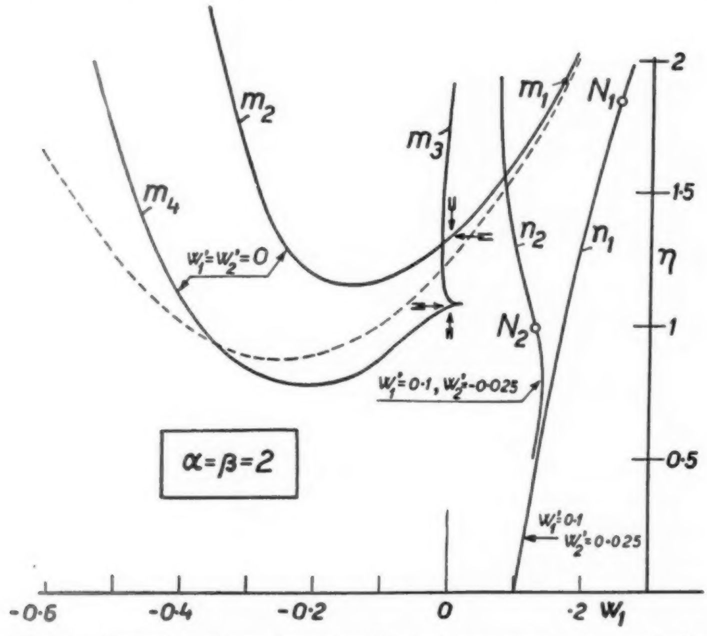

FIG. 6. Panel subjected to uniform shortening. Shortening parameter η versus displacement component w_1.

represents the virtual work done by the forces applied to the panel for the displacements corresponding to the actual deformation. It may be shown that, according to the condition of uniform shortening, this gives

$$\sigma = \frac{1}{2abt'} \int_0^{2a} N \, dx$$

and finally

$$\sigma/\sigma_c = \eta + 3f_3/\beta^2 \tag{29}$$

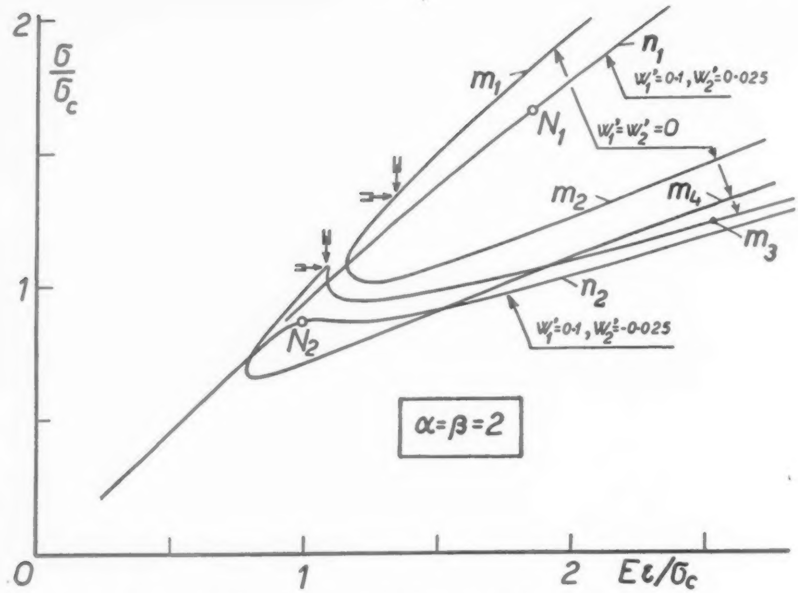

FIG. 7. Panel subjected to uniform shortening. Stress-strain diagram.

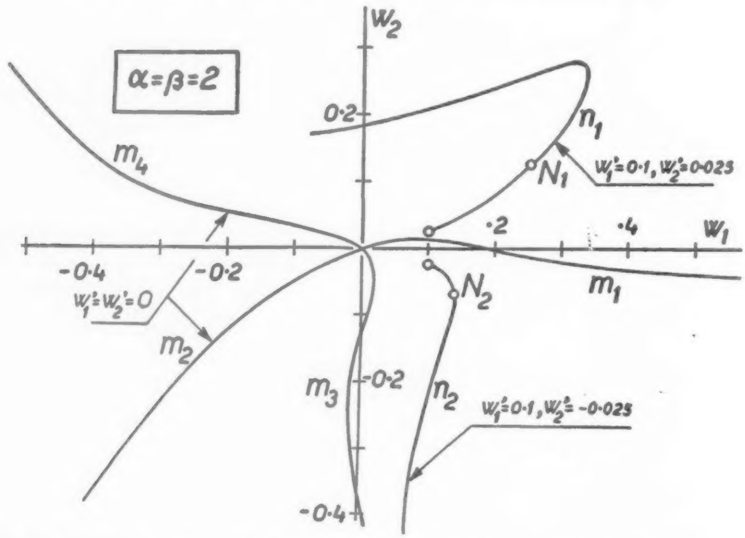

FIG. 8. Panel subjected to uniform shortening. Displacement components w_2 versus w_1.

Numerical calculations were made for $\alpha = \beta = 2$ and the results are plotted in Figs. 6, 7 and 8; corresponding points and lines in the figures are marked with the same letters.

The curves m_1m_2 and m_3m_4 in the figures apply to a panel with perfect initial form, and the arrows on the curves in Figs. 6 and 7 denote points where the deflection components w_1 and w_2 are both zero. Of the four branches of the curves starting from these

points, only m_1 gives a favorable characteristic. Favorable behaviour of the panel can only be expected for initial deformations whose representative points fall on or near the branch m_1 in Fig. 8. This indicates that favorable initial deformations are directed outwards, with maximum values near the center of the panel.

Curves n_1 and n_2 depart from the favorable path m_1 and this phenomenon is similar to that observed for a complete shell. Curve n_2 departs immediately towards the large deflection regime with little increase in the end load. However, the initial part of the

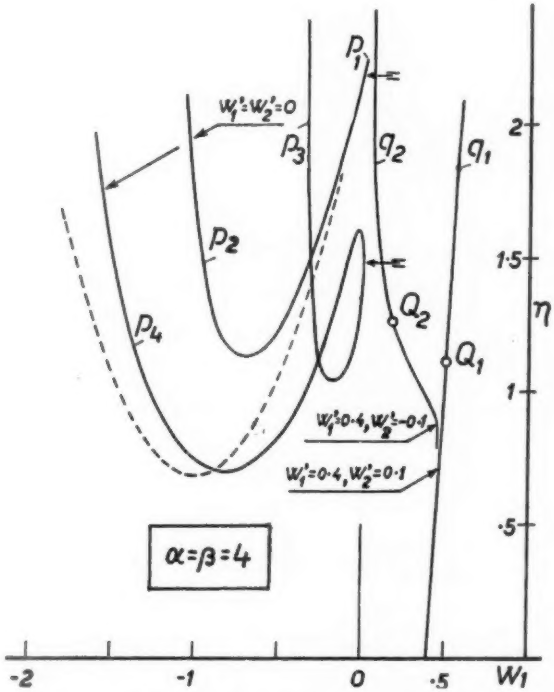

FIG. 9. Panel subjected to uniform shortening. Shortening parameter η versus displacement component w_1.

n_1 curve corresponds to increasing loads, as shown in Fig. 7. After reaching a certain point, the curve in Fig. 8 suddenly veers in the direction of the m_4 branch. This phase occurs with decrease of load but, before it occurs, the load reaches such large values that stresses may be obtained in excess of the elastic limit of the material. For this reason, the portion of the n_1 curve in Fig. 7 which corresponds to decreasing loads is not shown.

Similar results were obtained from calculations made for $\alpha = \beta = 4$, the various graphs being given in Figs. 9, 10 and 11. From the points corresponding to $w_1 = w_2 = 0$, which are indicated by arrows in Figs. 9 and 10, the favorable path is p_1. Line q_1 suddenly veers towards p_4 but, before it does so, very high loads are attained. Line q_2 soon approaches the branch p_3 in Fig. 11 and indicates unfavorable initial deformations.

The dotted curves in Figs. 6 and 9 correspond to $w_2 = f_2 = 0$, when the second and fourth of Eqs. (28) are left out.

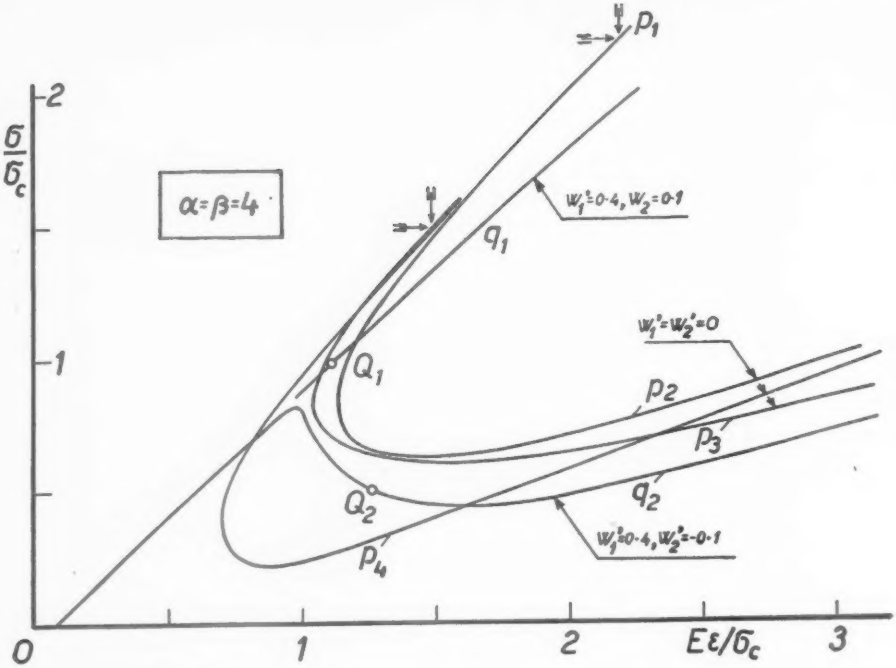

FIG. 10. Panel subjected to uniform shortening. Stress-strain diagram.

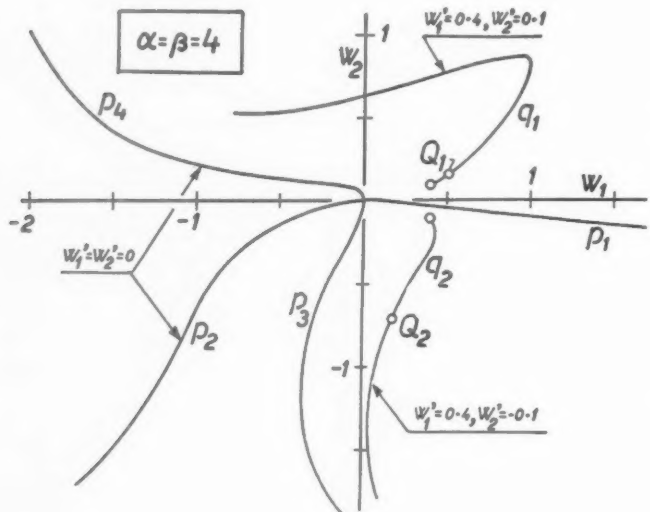

FIG. 11. Panel subjected to uniform shortening. Displacement components w_2 versus w_1 .

(b) Panel subjected to uniform longitudinal stress

The new boundary condition, replacing the one expressed by Eq. (26), is

$$\frac{\partial^2 f}{\partial y^2} = -2\eta t/R \qquad \text{for} \qquad x = 0 \qquad \text{and} \qquad x = 2a.$$

The parameter η represents here the ratio of the applied uniform stress to the critical stress obtained from the linear theory for a complete cylindrical shell. The new con-

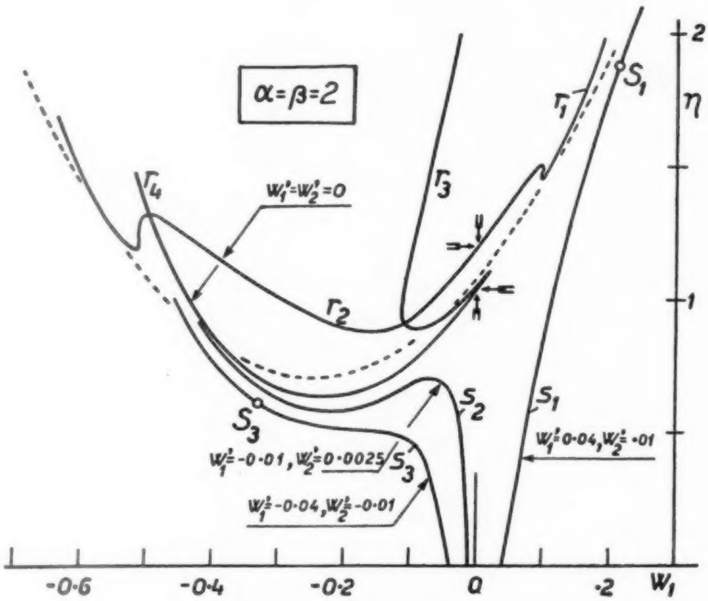

Fig. 12. Panel subjected to uniform longitudinal stress. Stress parameter η versus displacement component w_1.

dition, as well as those expressed by Eqs. (22), (23) and (24) are found to be satisfied by writing

$$\begin{aligned} f/3\pi t^2 &= [f_1(1 - \cos \xi) + f_2(1 - \cos 2\xi)] \sin \varphi - \eta y^2/3\pi R t, \\ w/3\pi t &= [w_1(1 - \cos \xi) + w_2(1 - \cos 2\xi)] \sin \varphi. \end{aligned} \tag{30}$$

Proceeding as before, we obtain

$$\begin{aligned} D_1 f_1 + 2\frac{\alpha^2}{\beta^2}(f_1 + f_2) &= -\beta^2 w_1 - 12 w_1^2 - 16 w_2^2 + 4 w_1 w_2 , \\ D_2 f_2 + \frac{\alpha^2}{2\beta^2}(f_1 + f_2) &= -\beta^2 w_2 - 12 w_2^2 - \frac{1}{2} w_1^2 - 8 w_1 w_2 , \end{aligned} \tag{31}$$

$$2\eta\beta^2 w_1 = D_1 w_1 + 2\frac{\alpha^2}{\beta^2}(w_1 + w_2) - (\beta^2 + 24w_1 - 4w_2)f_1 - (32w_2 - 4w_1)f_2 + c_1 \ ,$$

$$2\eta\beta^2 w_2 = D_2 w_2 + \frac{\alpha^2}{2\beta^2}(w_1 + w_2) - (\beta^2 + 24w_2 + 8w_1)f_2 - (8w_2 - w_1)f_1 + c_2 \ ,$$

with the same notation previously used.

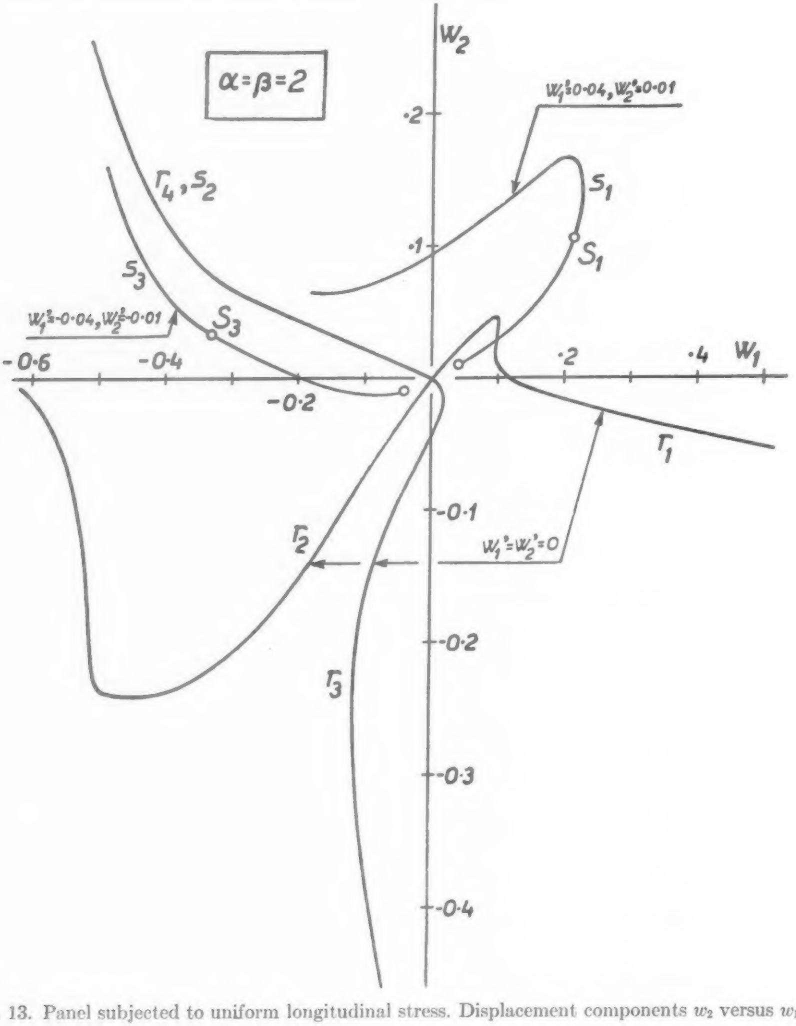

FIG. 13. Panel subjected to uniform longitudinal stress. Displacement components w_2 versus w_1 .

Calculations were made for $\alpha = \beta = 2$ and the results plotted in Figs. 12 and 13. Stress-strain diagrams were not plotted since the end displacement varies with the particular generator chosen.

The conclusions drawn for a panel with uniform shortening also apply to a panel with uniform longitudinal stress. The arrows in Fig. 12 show the points corresponding to zero values of w_1 and w_2 for a panel without initial deformations. Branches r_1, r_2, r_3 and r_4 starting from these points correspond to the branches starting from the origin in Fig. 13.

Curves r_4 and s_2 in Fig. 13 are practically coincident. The dotted curve in Fig. 12 is obtained equating w_2 and f_2 to zero and disregarding the second and fourth of Eqs. (31).

Conclusions

(a) The cylindrical shell under axial compression

An approximate analysis has been made of the behaviour of an axially loaded cylindrical shell using a four-term expression for the stress function and also for the displacement function. However, numerical calculations were generally made using three terms in

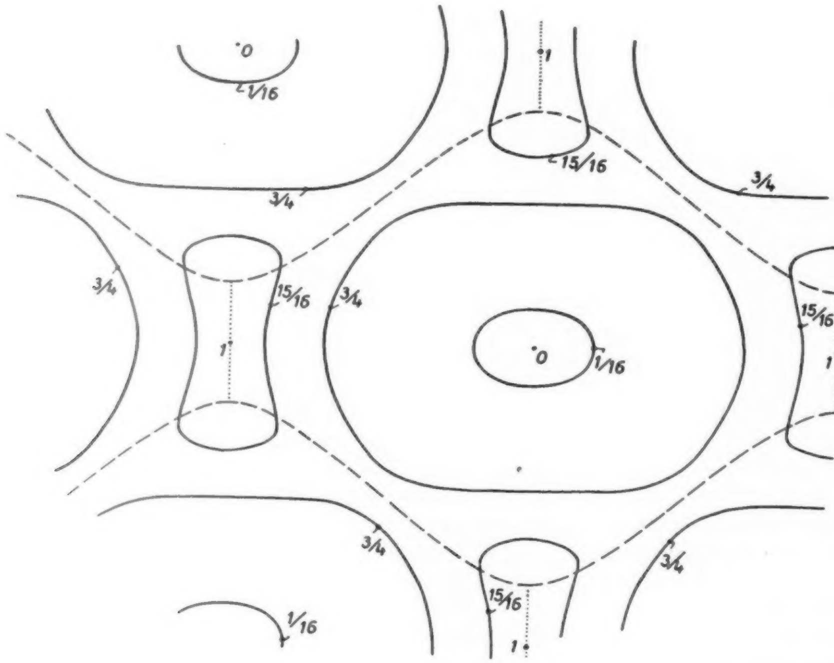

Fig. 14. Lines of equal deflection of wave surfaces, according to eq. (10), for $w_3 = w_4 = 0$, $w_1 = 3.2w_2$.

the expressions for the stress functions and two in the expressions for the displacements. The two terms retained are of the form sin $(\pi x/a)$ sin $(\pi y/b)$ and cos $(2\pi x/a)$, since previous experience has shown (2) that these terms are of greatest importance in the first phase of the buckling process.

Three modes of buckling appear possible for a cylinder with perfect initial form.

The most unfavorable one corresponds to the lower branch 0B of the corresponding curve in Fig. 1. In this case, the ratio of the amplitudes of the displacement components is about 4. The generators of the cylinder which contain the crests of the waves in the buckled form take a shape similar to that given by the curve (0) in Fig. 4 but with opposite sign, that is, with flat tops directed outwards from the cylinder and sharp tops directed inwards.

The curves formed by the intersection of the deflected surface with coaxial cylinders of various radii are shown in Fig. 14;[5] they refer to the case when the ratio of the amplitudes is 3.2. The axis of the cylinder is vertical in Fig. 14 and it is evident that the outwardly directed crests are elongated in the direction of the axis. This phenomenon is often noticed in buckled specimens.

The favorable type of buckling whose stress-strain characteristics are represented by the branch 0A of the curve in Fig. 1, has generators of the form given by curve (0) in Fig. 4. If this type of deformation could be ensured, the behaviour of a cylindrical shell under axial load would be practically unaffected by buckling, and the stress-strain curve would continue to follow the line representing Hooke's law.

The third type of buckling corresponds to an axially symmetrical configuration and this would be represented on the stress-strain diagram in Fig. 1 as a horizontal line through the point 0.

When the second deflection component in Eq. (7) is initially positive (that is, the inward crests are sharper than the outward ones), under axial loading, the deformations rapidly approach the most unfavorable form, no matter what the magnitudes of the initial deformations may be. The stress-strain curves for various initial deformations are given in Fig. 1.

When the second component in Eq. (7) is initially negative, it might seem possible to produce a favorable mode of buckling. However, a more accurate analysis has shown this to be impossible. If the first component is larger than the second, the magnitude of the latter is further decreased by axial loading until it reverses its sign and gives rise to the unfavorable mode of buckling. This is indicated by the curves c and d in Fig. 2. If the first component is the smaller, the second component could increase under load and lead to axially symmetrical buckling. However, this could only occur if planes normal to the axis of the cylinder containing the maxima of the first component, also contained the maxima of the second component. In reality this perfect coincidence never occurs. Then, as is shown by the analysis based on Eqs. (20), a rapid relative shift of the two components occurs. When the position of one component has changed by half a wavelength with respect to the other, the deflected form is practically inverted as indicated by the various curves in Fig. 4. The stress-strain curve for this case is still unfavorable.

It may therefore be concluded that, irrespective of the initial deformations existing in a cylindrical shell, buckling will always follow an unfavorable mode, with the stress-strain curve rapidly approaching that pertaining to the lower branch for a shell of perfect initial form. With cylinders having initial deformations of wavelength about $2\pi\sqrt{Rt}$, it is impossible to improve the stress-strain characteristics.

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[5]The radial distance from the intersection to the internally tangent cylinder is given on each curve as a fraction of the difference in radii of the externally and internally tangent cylinders. The dotted curve is the locus of points where the tangent plane is parallel to the axis of the cylinder.

For wavelengths greater than $2\pi\sqrt{Rt}$, it may be shown that, at buckling, the stress-strain curves start from a point for which $\eta > 1$ but that they fall rapidly to low minima. Such initial deformations are therefore dangerous.

The possibility of using initial deformations to improve the stress-strain characteristics of cylinders is only feasible for wavelengths considerably smaller than $2\pi\sqrt{Rt}$. Such wavelengths might stiffen the shell and discourage the buckling. This possibility is best investigated experimentally.[6]

(b) The panel under axial compression

The compressed cylindrical panel was analyzed assuming, as to radial displacements, that the curved sides were clamped and that the straight sides were simply supported. The expression chosen for the radial displacements contains two terms and Fig. 15

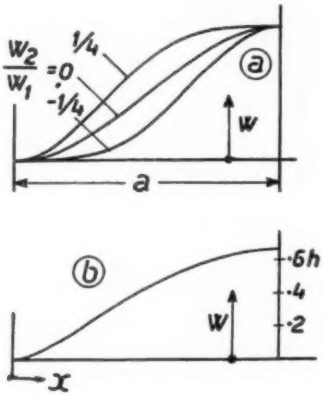

FIG. 15. Deformation of mid-generator, for $a = b$; (a) according to eq. (32), (b) according to eq. (45).

shows the forms assumed by the middle generator for three ratios of the maximum values of these terms.

Calculations were made for two special cases: (a) panel subjected to constant axial shortening and, (b) panel subjected to constant axial stress. The conclusions drawn from the two cases are the same. As for a complete cylindrical shell, the deformations tend towards the unfavorable modes but it is possible for the critical load of the linear theory to be exceeded before the beneficial effects of certain initial deformations are nullified. The approximate analysis given in the Appendix shows that favorable initial deformations may be obtained by initially bending a panel to a radius which is smaller than that at which it is tested. This may be seen from the stress-strain curves n_1 and q_1 in Figs. 7 and 11 respectively, which correspond to the curve $w_2/w_1 = 1/4$ in Fig. 15(a); Welter's experimental results also indicate that a continuous and regular stress-strain curve is possible. With proper experimental verification, the foregoing analysis will serve to indicate how the behaviour of a compressed panel may be improved.

[6]For short cylinders, the barrel shape might be beneficial.

Acknowledgement

This investigation was carried out as a part of the Cornell Aeronautical Laboratory program of internal research, and the author wishes to express his appreciation to Dr. A. F. Donovan, Head of the Aero-Mechanics Department of the Laboratory who suggested this problem for investigation. The author is greatly indebted to Prof. H. D. Conway of Cornell University for his accurate revision and linguistic correction of the paper.

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Appendix

Initial deformations caused by clamping a panel in cylindrical grips.

In Welter's tests (3) some panels were bent to various radii and then tested in grips of smaller radius. In order to estimate the initial deformations likely to have occurred in those conditions, we consider the following schematic problem.

A rectangular panel whose sides are $x = 0$, $x = 2a$, $y = 0$ and $y = b$, is initially bent to a shape $w^* = h \sin \varphi$ and is then constrained such that the following boundary conditions are imposed

$$w = \frac{\partial w}{\partial x} = 0 \qquad \text{for} \qquad x = 0,\, 2a; \qquad w = \frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{for} \qquad y = 0,\, b.$$

Making use of Eq. (5) and assuming small displacements, we write

$$\nabla^4(w - w^*) = 0.$$

According to the boundary conditions, the solution takes the form:

$$w = \frac{h \sin \varphi}{\sinh \chi_0 \cosh \chi_0 + \chi_0} [(\sinh \chi_0 + \chi_0 \cosh \chi_0)(\cosh \chi_0 - \cosh \chi) + (\chi \sinh \chi - \chi_0 \sinh \chi_0) \sinh \chi_0], \tag{32}$$

where

$$\chi = \pi(x - a)/b, \qquad \chi_0 = \pi a/b.$$

This equation gives an estimate of the initial deformations occurring when a panel, initially bent to a cylindrical surface of small curvature, is clamped in grips, also of

small curvature, but with a camber different by h from that of the panel. The deflections obtained for $a = b$, for the generator $y = b/2$ are plotted in Fig. 15(b).

The condition encountered in Welter's tests were, of course, somewhat different from the above. For example, the grips in the tests gave a large camber to the panels and this would require the use of Eqs. (5) in their complete form. Rotation of the straight edges was also restricted by the toothed guides. For these and other reasons, it is obvious that the above is not a rigorous solution. However it will give an estimate of the initial deformations which may be expected.

ON THE PLANE PROBLEM OF A PERFECT PLASTIC BODY[1]

BY

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In the equilibrium problem of the isotropic ideal plane plastic body it is assumed that the state of stress is restricted to a one-dimensional variety $F(\sigma_1, \sigma_2) = 0$ where $\sigma_i (i = 1, 2)$ denote the principal stresses; moreover, the two equilibrium conditions hold. This problem may be derived in several ways from the complete three-dimensional problem of the perfect plastic body; in particular the assumption $\sigma_3 = \sigma_z = 0, \partial \cdots/\partial z = 0$ leads to it without the need for consideration of any three-dimensional relation between strain-rates and stresses. However, as long as $F(\sigma_1, \sigma_2)$ is not specified the relation of the plane problem to the three-dimensional problem need not be considered.

Important particular assumptions for the yield condition, $F(\sigma_1, \sigma_2) = 0$, are the v. Mises condition of constant shear energy and the Saint Venant condition of constant maximum shearing stress. For the first, F is given by

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = 4K^2, \tag{1}$$

the second states that $|\tau|_{\max} = |\sigma_1 - \sigma_2|/2 = K$, if σ_1 and σ_2 have opposite signs, while otherwise either $\sigma_1 = 2K$, or $\sigma_2 = 2K$. Hence

$$\begin{aligned} |\sigma_1 - \sigma_2| &= 2K, && \text{if} \quad \sigma_1\sigma_2 \leqq 0, \\ &= 4K - |\sigma_1 + \sigma_2|, && \text{if} \quad \sigma_1\sigma_2 \geqq 0. \end{aligned} \tag{2}$$

These forms of F are derived from corresponding three-dimensional conditions by means of the afore-mentioned assumption of "plane stress", $\sigma_3 = 0, \partial \cdots/\partial z = 0$.

The problem most widely investigated is that with the yield condition $(\sigma_1 - \sigma_2)^2 = 4K^2$ as in the first Eq. (2). It is distinguished by its comparative mathematical simplicity; moreover, this same yield condition appears in "plane strain", as both v. Mises' and Saint Venant's condition.

The general plane problem defined by the two equilibrium equations and some yield condition has been considered by Sokolovsky [6a, b] for the two conditions (1) and (2); and for general yield conditions by v. Mises [4a], Geiringer [1], and the Brown University group, (see P. G. Hodge [2], Chap. VIII). As in the case of "plane strain" one can—in several ways—derive from the basic equations a *linear* system by a change of variables, somehow analogous to the Chaplygin transformation in the theory of compressible fluids; the plane of the new independent variables called "stress graph" by v. Mises, corresponds then to the hodograph plane, the yield condition to the "adiabatic" condition. It is intended to follow up this analogy in several directions.

First we establish *linear differential equations of second order* for certain magnitudes which may serve as coordinates in the physical plane; such an equation is amenable to well-known analytic methods. The coefficients depend, of course, on the yield condition;

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[1]Received Oct. 13, 1950. The main results of this paper have been communicated in a talk given at the Second Symposium on Plasticity, Providence, April 1949, and later briefly summarized in three abstracts of papers presented by title at the October 1949 meeting of the A.M.S. [1].

we consider particularly conditions (1) and (2) and the new "parabola condition" recently proposed [4a] by v. Mises. Next the *characteristics* are investigated, in general, and for the afore-mentioned yield conditions; here most of our results while definitely more simple are essentially identical with corresponding results of Sokolovsky [6a], v. Mises [4a], Hodge [2]. Finally, *simple wave solutions* are considered. The stress distributions are determined and discussed; cross characteristics and lines of principal stress are computed. In each of the indicated directions one can go much further, this paper thus being of an introductory character; solutions of concrete problems have not been included.

1. The stress graph. Following v. Mises we express the plasticity condition in terms of an appropriate parameter, s:

$$F(\sigma_1\,,\,\sigma_2) = 0, \qquad \sigma_1 = \sigma_1(s), \qquad \sigma_2 = \sigma_2(s). \tag{3}$$

Let us assume from now on that $\sigma_2 \geqq \sigma_1$. It is often convenient to take for s the average stress $(\sigma_1 + \sigma_2)/2K$. Denoting by θ the angle between the positive x direction and the ξ direction, i.e., the (positive) first principal direction, we take s and θ as new variables which determine the stress tensor at each point. The plane in which s and θ are polar coordinates is called *stress graph* by v. Mises. Denote by $\partial \cdots/\partial\xi$ and $\partial \cdots/\partial\eta$ the directional derivatives in the first and second principal direction, respectively, then equilibrium conditions are equivalent to the system (3):

$$\frac{\partial\sigma_1}{\partial\xi} = (\sigma_2 - \sigma_1)\frac{\partial\theta}{\partial\eta}, \qquad \frac{\partial\sigma_2}{\partial\eta} = (\sigma_2 - \sigma_1)\frac{\partial\theta}{\partial\xi}. \tag{4}$$

With the use of (3) and the abbreviations, $d\sigma_1/ds = \sigma_1'(s)$, etc.,

$$(\sigma_2 - \sigma_1)/\sigma_1' = f(s), \qquad (\sigma_2 - \sigma_1)/\sigma_2' = g(s), \tag{5}$$

(4) yields the *reducible* equations

$$\frac{\partial s}{\partial\xi} = f(s)\frac{\partial\theta}{\partial\eta}, \qquad \frac{\partial s}{\partial\eta} = g(s)\frac{\partial\theta}{\partial\xi}. \tag{6}$$

At every point where the Jacobian $j = \partial(s,\,\theta)/\partial(\xi,\,\eta)$ is different from zero in (6) the dependent and independent variables may be interchanged, to yield v. Mises' linear equations

$$\frac{\partial\eta}{\partial\theta} = f(s)\frac{\partial\xi}{\partial s}, \qquad \frac{\partial\xi}{\partial\theta} = g(s)\frac{\partial\eta}{\partial s}. \tag{7}$$

2. Linear differential equations of second order. In Eq. (6) ξ and η are not coordinates; it is, however, possible to derive from (7) in various ways systems of equations for quantities which may serve as coordinates in the physical plane. If, e.g.,

$$X = x\cos\theta + y\sin\theta, \qquad Y = y\cos\theta - x\sin\theta, \tag{8}$$

we find the system of linear equations of first order:

$$\frac{\partial X}{\partial\theta} = g(s)\frac{\partial Y}{\partial s} + Y, \qquad \frac{\partial Y}{\partial\theta} = f(s)\frac{\partial X}{\partial s} - X. \tag{9}$$

From (9) we obtain the equations of second order:

$$\frac{\partial^2 X}{\partial \theta^2} - fg\,\frac{\partial^2 X}{\partial s^2} = -X + \frac{\partial X}{\partial s}\,(f'g + f - g),$$
$$\frac{\partial^2 Y}{\partial \theta^2} - fg\,\frac{\partial^2 Y}{\partial s^2} = -Y + \frac{\partial Y}{\partial s}\,(g'f + f - g). \tag{10}$$

These equations[1] are hyperbolic, or elliptic, according to whether $fg \gtrless 0$ (see (5)) or:

$$\frac{d\sigma_1}{ds}\,\frac{d\sigma_2}{ds} \gtrless 0; \tag{11}$$

they are parabolic if either fg or its reciprocal value equals zero. If a solution $X(s, \theta)$ of (10^1) has been found, the corresponding $Y(s, \theta)$ follows from (9^2) and the coordinates x, y follow from (8) in terms of s and θ without further integration; thus s and θ are defined in terms of x and y.

Next, we satisfy the first Eq. (9) by a function $\psi(s, \theta)$ such that

$$\frac{\partial \psi}{\partial s} = \frac{X}{g(s)k(s)}, \qquad \frac{\partial \psi}{\partial \theta} = \frac{Y}{k(s)}, \tag{12}$$

where k satisfies the condition $g(s)k'(s) + k(s) = 0.$ (12′)
Substituting ψ into the second Eq. (9), we find

$$\frac{\partial^2 \psi}{\partial \theta^2} - fg\,\frac{\partial^2 \psi}{\partial s^2} = \frac{\partial \psi}{\partial s}\,(fg' - f - g). \tag{13}$$

Also with

$$\frac{\partial \varphi}{\partial s} = \frac{Y}{f(s)h(s)}, \qquad \frac{\partial \varphi}{\partial \theta} = \frac{X}{h(s)}, \qquad f(s)h'(s) - h(s) = 0, \tag{14}$$

the equation

$$\frac{\partial^2 \varphi}{\partial \theta^2} - fg\,\frac{\partial^2 \varphi}{\partial s^2} = \frac{\partial \varphi}{\partial s}\,(gf' + g + f) \tag{15}$$

follows in the same manner.

For finding integrals of (10) or (13), or (15) two principal methods are available: (a) Bergman's operator method may be adapted to our problem; (b) the classical way of expansion in series seems promising. In these ways regular solutions as well as solutions with appropriate singularities may be obtained. Because of the linearity of (10) solutions may be superimposed. On the other hand, the difficulties encountered in actual boundary value problems, where the boundary conditions are given in the physical plane, are well known.

3. Particular yield conditions. We consider v. Mises' "quadratic" condition (1), the the "hexagonal" condition of Saint Venant, and v. Mises' new "parabola condition".

[1]Equations (10) are simpler than Eq. (19) of v. Mises [4a] because X and Y rather than x and y have been chosen as dependent variables.

In the $\sigma_1 - \sigma_2$-plane (Fig. 1) the *Mises limit* (1) is represented by the ellipse

$$s = \frac{1}{2K}(\sigma_1 + \sigma_2), \qquad (-2 \leqq s \leqq +2),$$

$$\sigma_1 = K\left[s \pm \left(\frac{4 - s^2}{3}\right)^{1/2}\right], \qquad \sigma_2 = K\left[s \mp \left(\frac{4 - s^2}{3}\right)^{1/2}\right], \tag{16}$$

where the upper signs apply to the right, the lower to the left part of the ellipse. Then

$$\sigma_1'\sigma_2' = \frac{4K^2}{3}\frac{3 - s^2}{4 - s^2}, \qquad fg = \frac{(\sigma_2 - \sigma_1)^2}{\sigma_2'\sigma_1'} = \frac{(4 - s^2)^2}{3 - s^2}. \tag{17}$$

It is seen that $fg > 0$ for $s^2 < 3$, $fg < 0$ for $s^2 > 3$ while $fg \to \pm\infty$ as $s^2 \to 3$. Since we assume $\sigma_2 \geqq \sigma_1$ we are concerned merely with the left part of the ellipse; the hyperbolic case corresponds to the interval $-\sqrt{3} < s < +\sqrt{3}$; to $s = +\sqrt{3}$ and $s = -\sqrt{3}$

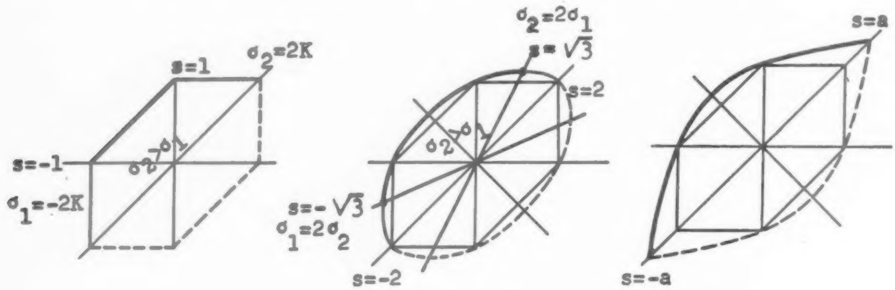

FIG. 1. Yield conditions.

correspond the "sonic points" $\sigma_2 = 2\sigma_1 = 4K/\sqrt{3}$ and $\sigma_1 = 2\sigma_2 = -4K/\sqrt{3}$. In the physical plane the "sonic line" separates the domain of hyperbolic solutions from the elliptic domain; its image in the stress graph is the circle about the origin with radius $s = \sqrt{3}$. The representation of the "hyperbolic part", for $\sigma_2 > \sigma_1$ is:

$$\sigma_1 = K\left[s - \left(\frac{4 - s^2}{3}\right)^{1/2}\right], \qquad \sigma_2 = K\left[s + \left(\frac{4 - s^2}{3}\right)^{1/2}\right], \qquad |s| < \sqrt{3}. \tag{16'}$$

The complete *Saint-Venant limit* (2) is represented by a hexagon inscribed in the ellipse (16). With $s = (\sigma_1 + \sigma_2)/2K$, the representation of the "hyperbolic part" of this limit is

$$\sigma_1 = K(s - 1), \qquad \sigma_2 = K(s + 1), \qquad |s| < 1 \tag{18}$$

and

$$\sigma_1'\sigma_2' = K^2, \qquad fg = 4. \tag{19}$$

The vertical and horizontal lines of the hexagon correspond to parabolic domains, where either σ_1 or σ_2 are constant.

The *"parabola limit"* consists of two branches of parabolas passing through four corners of the Saint Venant hexagon:

$$\frac{\sigma_2 - \sigma_1}{K} = \pm\frac{1}{a}\left[a^2 - \left(\frac{\sigma_1 + \sigma_2}{2K}\right)^2\right], \tag{20}$$

where $a = 1 + \sqrt{2}$, the plus sign applying to the left, the minus to the right branch. The parametric representation for $\sigma_2 > \sigma_1$ is (with our parameter)

$$\sigma_1 = Ks - \frac{K}{2a}(a^2 - s^2), \qquad \sigma_2 = Ks + \frac{K}{2a}(a^2 - s^2), \qquad |s| < a \tag{21}$$

and

$$f = a - s, \qquad g = a + s, \qquad fg = a^2 - s^2. \tag{22}$$

It is seen that, with the only exception of the points $s = \pm a$, where $fg = 0$, the problem is everywhere hyperbolic.

As an application let us consider the second order equations of the preceding section for the *parabola limit*, transforming them at the same time to standard forms. With

$$\frac{ds}{du} = (a^2 - s^2)^{1/2}, \qquad s = a \sin u, \qquad u = \arcsin s/a$$

and putting $X(s, \theta) = \overline{X}(u, \theta)$, and $A(u) = (1 + 2 \sin u)/\cos u$, Eq. (10) becomes

$$\frac{\partial^2 \overline{X}}{\partial \theta^2} - \frac{\partial^2 \overline{X}}{\partial u^2} + \overline{X} \cdot A(u) = 0, \qquad |u| < \frac{\pi}{2}.$$

Next with $p(u) = [\cos u(1 - \sin u)]^{-1/2}$, where $Ap - 2p' = 0$, it assumes the form

$$\frac{\partial^2 Z}{\partial \theta^2} - \frac{\partial^2 Z}{\partial u^2} + Z \frac{1 + 2 \sin u}{4 \cos^2 u} = 0. \tag{10a}$$

This is an equation in standard form which may be treated in various manners.

Also, with $\psi(s, \theta) = \overline{\psi}(u, \theta)$, Eq. (13) takes the form

$$\frac{\partial^2 \overline{\psi}}{\partial \theta^2} - \frac{\partial^2 \overline{\psi}}{\partial u^2} + \frac{\partial \overline{\psi}}{\partial u} \cdot \frac{1}{\cos u} = 0, \tag{13a}$$

and putting $\overline{\psi}(u, \theta) = r(u) R(u, \theta)$ where

$$r(u) = \left(\frac{1 + \sin u}{\cos u}\right)^{1/2}, \qquad 2r' \cos u - r = 0,$$

we obtain

$$\frac{\partial^2 R}{\partial \theta^2} - \frac{\partial^2 R}{\partial u^2} + R \frac{1 - 2 \sin u}{4 \cos^2 u} = 0. \tag{13b}$$

Note the difference between (10a) and (13b).

As a second application, we give an example of particular solutions of Eqs. (13). Set

$$\psi(s, \theta) = P(s) Q(\theta). \tag{23}$$

One obtains in the usual way, denoting by λ a constant, and using (22):

$$(a^2 - s^2) P'' - (a + s) P' - \lambda P = 0.$$

Introducing the new variables u and U by

$$s = a(2u - 1), \qquad \frac{dP}{ds} = \frac{1}{2a} \frac{dU}{du}, \qquad \frac{d^2 P}{ds^2} = \frac{1}{4a^2} \frac{d^2 U}{du^2},$$

we find

$$u(1 - u) \frac{d^2U}{du^2} - u \frac{dU}{du} - U = 0, \tag{24}$$

which is a hypergeometric equation. Solutions of (24) are well known; with $Q = C \sin n(\theta - \theta_0)$, we may thus find solutions of the form (23), which may be superimposed.

4. Characteristics. The characteristics of the reducible equations (6) are easily found. If φ denotes the angle of a characteristic direction with the ξ-direction, we find by standard methods:

$$\tan^2 \varphi = \frac{f}{g}, \qquad \tan \varphi = \frac{d\eta}{d\xi} = \pm\left(\frac{f}{g}\right)^{1/2}. \tag{25}$$

In the hyperbolic domain, f and g have the same sign; thus, at each point of this domain, there are two distinct directions making equal angles with the ξ-direction. We

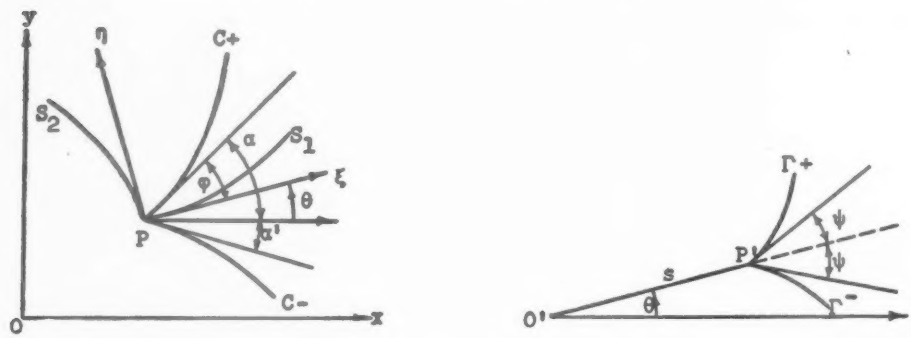

Fig. 2. Characteristics in physical plane and in stress plane.

denote by C^+, (C^-) the characteristics making the angles $+\varphi$, $(-\varphi)$ with the ξ-direction, and by

$$\alpha = \theta + \varphi, \qquad \alpha' = \theta - \varphi \tag{26}$$

the angles of C^+ and C^- with the x-axis (Fig. 2). Then, with dy/dx for the direction coefficient of a characteristic, we have

$$\frac{dy}{dx} = \tan (\theta \pm \varphi) = \frac{\tan \theta \pm (f/g)^{1/2}}{1 \mp \tan \theta \cdot (f/g)^{1/2}}, \tag{26'}$$

the upper (lower) sign applying to C^+ (to C^-). If a solution $s = s(xy)$, $\theta = \theta(x, y)$ is introduced into (26'), this becomes an ordinary differential equation for the determination of the characteristic curves.

Note also that, according to (3), (5), and (25) with $F_i = \partial F/\partial \sigma_i$,

$$\tan^2 \varphi = \frac{\sigma_2'}{\sigma_1'} = \frac{d\sigma_2}{d\sigma_1} = -\frac{F_1}{F_2}, \tag{25'}$$

depending merely on the yield condition: $F(\sigma_1, \sigma_2) = 0$. For the "quadratic" condition,

for example, this gives $\tan^2 \varphi = (\sigma_2 - 2\sigma_1)/(2\sigma_2 - \sigma_1)$, showing that along the "sonic line" φ equals 0 or $\pi/2$.

From the general definition of characteristics it follows that a relation exists between the derivatives of the dependent variables along a characteristic direction. To find this, multiply the first term in the first Eq. (6) by $\cos \varphi$, the second by $(g/f)^{1/2} \sin \varphi$; in the second Eq. (6) multiply the first term by $\sin \varphi$, the second by $(f/g)^{1/2} \cos \varphi$ and add.[1] Thus,

$$\frac{\partial s}{\partial \xi} \cos \varphi + \frac{\partial s}{\partial \eta} \sin \varphi = (fg)^{1/2} \cdot \left(\frac{\partial \theta}{\partial \xi} \cos \varphi + \frac{\partial \theta}{\partial \eta} \sin \varphi \right).$$

Denoting by $\partial \cdots / \partial l$ differentiation in a characteristic direction, we obtain the desired relations:

$$\frac{\partial s}{\partial l^+} = (fg)^{1/2} \frac{\partial \theta}{\partial l^+}, \qquad \frac{\partial s}{\partial l^-} = -(fg)^{1/2} \frac{\partial \theta}{\partial l^-}. \tag{27}$$

These relations between the derivatives of s and θ along C^+ or C^- do not depend on the validity of the transformation (7).

It follows from equation (10) or (15) that for the fixed characteristics Γ in the stress graph:

$$\left(\frac{d\theta}{ds} \right)^2 = (fg)^{-1}, \qquad \frac{d\theta}{ds} = \pm (fg)^{-1/2}. \tag{28}$$

Writing:

$$\int^s (fg)^{-1/2} \, ds = F(s) \tag{28'}$$

(with an adequate lower limit), we obtain the equations of the two families of fixed characteristics, Γ^+ and Γ^-:

$$\theta - F(s) = \text{const}, \qquad \theta + F(s) = \text{const}. \tag{28''}$$

Equation (28) may also be derived from (7) in the same way as (24) was derived from (6). Then (27) shows that (as always) the characteristics in the two planes which are mapped onto each other (here the x, y plane and the s, θ plane) correspond to each other, here C^+ to Γ^+, C^- to Γ^-. We see from (28) that Γ^+ and Γ^- make equal angles $+\psi$ and $-\psi$ with the radius vector and $\tan \psi = s d\theta/ds$ (Fig. 2).

We consider here in general the hyperbolic case; let us, however, just say a word about the parabolic case where $f/g = \sigma_2'/\sigma_1'$ is either zero or infinite. If $\sigma_2' = 0 (\sigma_1' = 0)$ the angle $\varphi = 0$, $(\varphi = \pi/2)$ and the one existing characteristic direction coincides with the direction of first (second) principal stress. In either case $\psi = 0$, $d\theta/ds = 0$. It follows from (28) that in a parabolic region the characteristics Γ in the stress graph are radii through O', and from (26') by integration that the characteristics in the physical plane form likewise a set of straight lines: $y = x \tan \theta + G(\theta)$ if $\sigma_2' = 0$.

5. Characteristics. Continuation. The Γ-curves in the stress graph have been investigated by v. Mises for the three yield conditions and sketched with s and θ as polar coordinates. The formulas are:

[1]In the hyperbolic case g/f is nowhere zero or infinite.

Quadratic condition (v. Mises)

$$\frac{d\theta}{ds} = \pm\frac{(3 - s^2)^{1/2}}{4 - s^2}$$

$$\pm\,\theta = \arctan\,[s(3 - s^2)^{-1/2}] - \frac{1}{2}\arctan\left[\frac{s}{2}(3 - s^2)^{-1/2}\right] + \text{const}, \tag{29}$$

and for the C- characteristics:

$$\tan\varphi = \pm\frac{(12 - 3s^2)^{1/2} - s}{2(3 - s^2)^{1/2}}. \tag{30}$$

At the "sonic points" tan φ is zero or infinite and $\psi = 0$. The characteristics (29) are congruent curves (like the epicycloids in our analogy). In rectangular coordinates, s and θ, a Γ^+ line is monotonously increasing e.g. from $-\pi/4$ to $\pi/4$, showing central symmetry, inflection points at $s = 0$, $s = \pm\sqrt{2}$, and $d\theta/ds = 0$ at $s = \pm\sqrt{3}$ (Fig. 3).

FIG. 3. $\theta = F(s)$.

Saint Venant condition. Here $d\theta/ds = \pm 1/2$. The Γ-characteristics in the hyperbolic area in the stress graph are ordinary spirals $\pm\theta = s/2 + \text{const}$, or straight lines in the rectangular coordinates s, θ (in the parabolic area the unique set consists of radii through O'); also $\tan\varphi = \pm 1$, $\alpha = \theta + \pi/4$, $\alpha' = \theta - \pi/4$; it is thus seen that the characteristics in the xy-plane coincide with the shear lines, a result well known in the classical theory.

Parabola condition.

$$\frac{d\theta}{ds} = \pm(a^2 - s^2)^{-1/2}, \qquad \theta = \pm\arcsin\frac{s}{a} + \text{const}. \tag{31}$$

These are sine-lines $s = \pm a \sin(\theta - \theta_0)$ if interpreted in rectangular coordinates and circles through the origin in polar coordinates. Here,

$$\tan\varphi = \pm\left(\frac{a - s}{a + s}\right)^{1/2}. \tag{32}$$

We finish this section by a *remark on the approximate solution* of the classical initial value problems in the hyperbolic domain. In the *Cauchy problem*, on a smooth arc of curve, K, values of s and θ are prescribed (with continuous second derivatives) so that

K has nowhere a characteristic direction, i.e. that (26′) does not hold. These data assure *uniqueness* of solution in the characteristic quadrangle determined by both, C^+ and C^-, at both endpoints of K. There *exists* certainly a solution in the neighborhood of K which on K takes on the prescribed values. It can be obtained approximately by using the fixed net of Γ-characteristics (adjacent to the image K' of K) in order to find the (s, θ) values for corresponding lattice points in the physical plane. In general, we may then be able to continue in this way until the whole characteristic quadrangle has been filled. The procedure may be refined in various ways.

For the *characteristic initial value problem* of reducible equations, like (6), the data which can be arbitrarily prescribed are not the same as in the better known linear case. (This difference is often not appreciated, see [2], p. 197.) In the reducible case any arbitrarily chosen curve $C : y = y(x)$ may be a characteristic, but along this C neither s nor θ may be arbitrarily prescribed, since along C one relation (26′) and one (28) (or rather (27)) must hold. More generally, for a curve $y = y(x)$ to be a characteristic, the three functions $y(x)$, $s(x)$, $\theta(x)$ must satisfy two relations: *one* (26′), and *one* (28).

For two intersecting characteristics, C^+, C^- the values of s and θ at the point of intersection must satisfy both relations (26′); the relations (28), which merely connect $d\theta$ and ds leave the values at one point unrestricted (or, in other words, the constants in

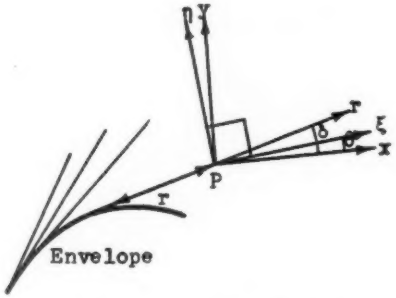

FIG. 4. Family of straight characteristics.

(28″) may be determined so that the four equations (26′) and (28″) are satisfied at the point of intersection). If in this way two intersecting characteristics are given, the solution in the corresponding quadrangle is uniquely determined, and it may be found step by step in a way analogous to that indicated above.

6. Simple waves. A simple wave region is a domain D in the physical plane mapped onto one single characteristics, say Γ_0^+, of the s, θ plane such that to each point on Γ_0^+ corresponds a straight line in D carrying constant values of s and θ (and consequently of σ_1, σ_2, θ). In such a "forward wave" the straight lines are the C^- characteristics, and $\theta - F(s)$ is constant for all (x, y) in D; all characteristics of the other family, the "cross characteristics", C^+, are mapped onto Γ_0^+. The Jacobian, j, of this mapping vanishes, of course. Similarly we define a "backward wave". Naturally, not all the above noted properties are needed for the definition of a simple wave. The notion includes as particular case that of *constant stress tensor* in D.

Here we start with the problem *of finding solutions of Eqs.* (6) *such that the state of stress is constant along each straight line of an arbitrary set of lines.* Denote by δ the angle

of such a straight line, L, with the x direction (Fig. 4). Since σ_1, σ_2, and θ remain constant along L, (4) yields

$$\sin(\theta - \delta)\frac{d\sigma_1}{d\delta} = (\sigma_2 - \sigma_1)\cos(\theta - \delta)\frac{d\theta}{d\delta},$$
$$\cos(\theta - \delta)\frac{d\sigma_2}{d\delta} = (\sigma_2 - \sigma_1)\sin(\theta - \delta)\frac{d\theta}{d\delta}. \tag{33}$$

By division, we obtain

$$\tan^2(\theta - \delta) = \frac{d\sigma_2}{d\sigma_1} = \frac{-F_1(s)}{F_2(s)} \tag{34}$$

as in (25). Next we multiply the first Eq. (33) by $\cos(\theta - \delta)$, the second by $\sin(\theta - \delta)$ and add:

$$\frac{d(\sigma_1 + \sigma_2)}{d\theta} = \frac{\sigma_2 - \sigma_1}{\sin(\theta - \delta)\cos(\theta - \delta)}. \tag{35}$$

From (35) and (34) we then find easily our Eq. (28). In fact,

$$d\theta = \frac{d(\sigma_1 + \sigma_2)}{\sigma_2 - \sigma_1}\sin(\theta - \delta)\cos(\theta - \delta) = \frac{(d\sigma_2 d\sigma_1)^{1/2}}{\sigma_2 - \sigma_1}$$

$$\frac{d\theta}{ds} = \frac{(\sigma_1'\sigma_2')^{1/2}}{\sigma_2 - \sigma_1} = (fg)^{-1/2}. \tag{35'}$$

Thus, if a condition (3) and a Γ_0^+ are chosen we find to each δ the s and θ by (28) and (34). If, however, we want to find *cross characteristics*, C^+, and *lines of principal stress* we have to consider not only δ but the whole set of straight characteristics. Let it be given in the form

$$y = \beta x + h(\beta), \tag{36}$$

where $\beta = \tan\delta = \tan(\theta - \varphi)$. The differential equation of the cross characteristics is $dy/dx = \tan(\theta + \varphi)$ and since θ and φ both depend in a given way on s, and s on β we have

$$\frac{dy}{dx} = \tan(\theta + \varphi) = k(\beta). \tag{37}$$

From (36) we derive $dy = x\,d\beta + \beta\,dx + h'd\,\beta$. Substituting this in (37), we obtain

$$\frac{dx}{d\beta} = \frac{x + h'(\beta)}{k(\beta) - \beta}. \tag{38}$$

Thus by means of quadratures x is obtained as a function of β; this together with (36) provides a parametric representation of the cross characteristics. For the *principal stress-lines* the procedure is the same except that $dy/dx = \tan\theta$ is used instead of (37) while the relation between s and β is the same as before. In case of a centered wave $h(\beta) = 0$.

It is well known that simple waves are adjacent to regions of constant state; this is

deduced immediately from the fact that a region of constant s, θ is mappeed onto a single point in the stress graph.

A simple wave can transform any state $s = s_1$, $\theta = \theta_1$ into another state $s = s_2$, $\theta = \theta_2$ provided either $\theta + F(s)$ or $\theta - F(s)$ (where $F(s)$ is defined in (28′)) has the same value for the first and second state. By combining a forward and a backward wave and inserting a uniform state in between, *any* s_2, θ_2 can be reached; in each wave the envelope of the straight characteristics can still be chosen in various ways.

7. Examples of simple waves. Consider the quadratic yield condition (1) and the forward wave, image of $\theta = F(s)$. Using (28′) and (29):

$$\theta = F(s) \equiv \arctan\,[s(3 - s^2)^{-1/2}] - \frac{1}{2}\arctan\left[\frac{s}{2}(3 - s^2)^{-1/2}\right]. \tag{39}$$

Here the C^- are straight lines, $\delta = \theta - \varphi$, and as $\sigma_2 > \sigma_1$, $\tan \varphi$ is given by (30) with $+$ sign. Hence

$$\delta = \theta - \varphi = \arctan \frac{s}{(3 - s^2)^{1/2}} - \frac{1}{2}\arctan \frac{s}{2(3 - s^2)^{1/2}} - \arctan \frac{(12 - 3s^2)^{1/2} - s}{2(3 - s^2)^{1/2}}.$$

To simplify, we introduce $t = +s(3 - s^2)^{-1/2}$ and obtain

$$\delta = \arctan t - \frac{1}{2}\arctan\frac{t}{2} - \arctan\left[\frac{1}{2}(4 + t^2)^{1/2} - t\right],$$

which reduces to

$$\delta = \arctan t - \frac{\pi}{4}, \qquad t = \tan\left(\delta + \frac{\pi}{4}\right) \tag{40}$$

or, reintroducing s,

$$s = \sqrt{3}\sin\left(\delta + \frac{\pi}{4}\right), \qquad \theta = F(s). \tag{41}$$

Here as s goes from $-\sqrt{3}$, to 0, to $+\sqrt{3}$, (t goes from $-\infty$, to 0, to $+\infty$), θ goes from $-\pi/4$, to 0, to $+\pi/4$ and δ from $-3\pi/4$ to $+\pi/4$ (see Fig. 3). Hence in a "complete" wave, δ varies by 180°. In actual problems a solution is given in such parts of the x, y plane only where the straight characteristics do not intersect; thus parts of complete waves may appear as solutions.

The image of the Γ_0^- characteristic $\theta + F(s) = 0$ is a backward wave with rectilinear C^+ lines and $\delta = \theta + \varphi = -[F(s) - \varphi]$. Hence this Γ^- wave is the reflected Γ^+ wave considered above and $\delta = \pi/4 - \arctan t$, etc. We see that in a foward wave as δ increases, the mean pressure as well as θ increase in a monotonous way. In a backward wave the opposite is true.

Next, we compute the *cross characteristics* for the backward wave $\theta = -F(s)$, $\delta = \pi/4 - \arctan t$. With notations as in (26), (36), and (37), we obtain:

$$\beta = \tan \delta = \tan\left(\frac{\pi}{4} - \arctan t\right) = \frac{1 - t}{1 + t}, \qquad t = \frac{1 - \beta}{1 + \beta}.$$

$$\tan(\theta - \varphi) = \tan \alpha' = -\frac{t^2 + t + 2}{t^2 - t + 2} = -\frac{\beta^2 + \beta + 2}{2\beta^2 + \beta + 1} = k(\beta).$$

We may then continue as explained above. For the particular case of a centered wave, where $h(\beta) = 0$, $\beta = y/x$, the differential equation of the cross characteristics becomes

$$\tan \alpha' = \frac{dy}{dx} = -\frac{y^2 + xy + 2x^2}{2y^2 + xy + x^2}, \tag{42}$$

which can be integrated easily. One finds, using polar coordinates, r, δ (Fig. 5)

$$r^2 \cos\left(\frac{\pi}{4} - \delta\right) = \text{const.} \tag{43}$$

Let us conclude this consideration of the quadratic condition by deriving a simple characterization of the stress tensor which holds if at each point of the wave we use a

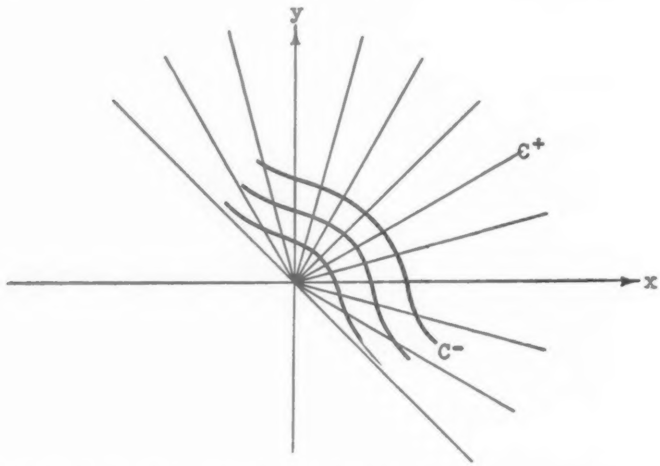

FIG. 5. Quadratic limit. Complete centered wave. Straight characteristics and cross characteristics.

coordinate system x', y' where x' is the direction of the straight line through this point, y' the perpendicular direction. The stress tensor is then given by $\sigma_{x'}$, $\sigma_{y'}$, $\tau_{x'y'}$. Denote by ν_1, ν_2 the two principal directions at a point and let $\sigma_{x'} = \sigma$, $\sigma_{y'} = \bar{\sigma}$, $\tau_{x'y'} = \tau$; then $(\nu\nu_1) = \delta - \theta = \varphi$ and we find

$$\sigma = \sigma_1 \cos^2 \varphi + \sigma_2 \sin^2 \varphi, \qquad \tau = (\sigma_2 - \sigma_1) \sin \varphi \cos \varphi. \tag{44}$$

Using primes to denote derivatives, we find from (1), (16′), (34):

$$\cos^2 \varphi = \sigma_1'/2K, \qquad \sin^2 \varphi = \sigma_2'/2K,$$

$$\begin{aligned} \sigma &= \frac{1}{2K}(\sigma_1\sigma_1' + \sigma_2\sigma_2') = \frac{1}{4K}\frac{d}{ds}(\sigma_1^2 + \sigma_2^2) = \frac{1}{4K}\frac{d}{ds}(\sigma_1\sigma_2) \\ &= \frac{1}{4K}\frac{d}{ds}\left[\frac{4K^2}{3}(s^2 - 1)\right] = \frac{2K}{3}s, \qquad \tau = \frac{2K}{3}(3 - s^2)^{1/2}. \end{aligned} \tag{45}$$

Also,

$$\bar{\sigma} = \frac{1}{2K}(\sigma_1\sigma_2' + \sigma_2\sigma_1') = \frac{1}{2K}\frac{d}{ds}(\sigma_1\sigma_2) = 2\sigma. \tag{46}$$

Hence under quadratic yield condition the following simple relations hold for the stress tensor all over in a simple wave region

$$\sigma^2 + \tau^2 = \frac{4}{3}K^2, \qquad \bar{\sigma} = 2\sigma. \tag{47}$$

Parabola limit. Using (31) and $s/a = t$, we consider the image of the Γ^- characteristic

$$\theta = -\arcsin t = \arccos t - \frac{\pi}{2}. \tag{48}$$

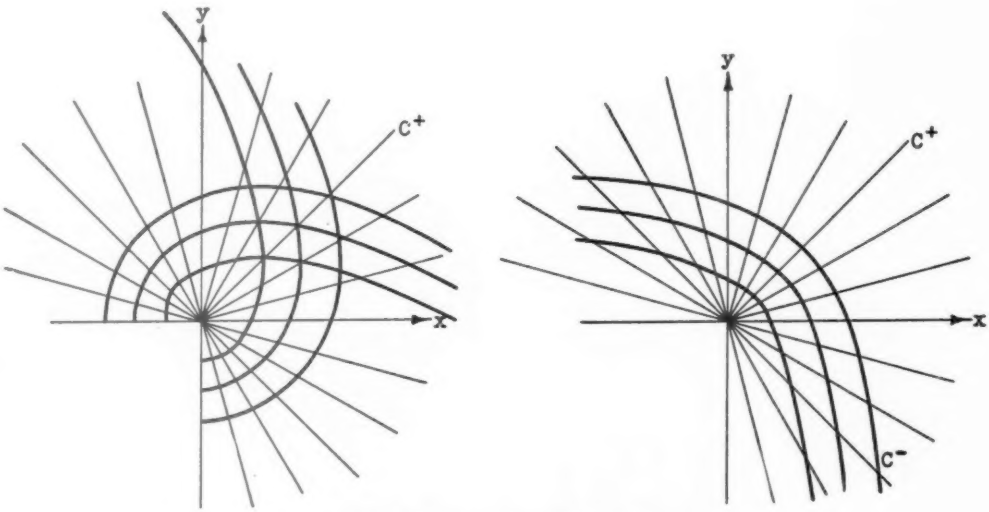

FIG. 6a. Parabola limit. Straight characteristics and cross characteristics.
FIG. 6b. Parabola limit. Lines of principal stress.

From

$$\tan\varphi = \left(\frac{a-s}{a+s}\right)^{1/2} = \left(\frac{1-t}{1+t}\right)^{1/2} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right), \tag{49}$$

we conclude that

$$\varphi = \frac{1}{2}\left(\theta + \frac{\pi}{2}\right), \qquad \delta = \varphi + \theta = \frac{3\theta}{2} + \frac{\pi}{4},$$

and thus:

$$\theta = \frac{2}{3}\left(\delta - \frac{\pi}{4}\right), \qquad s = a\cos\left(\theta + \frac{\pi}{2}\right) = a\cos\frac{\pi + 2\delta}{3}. \tag{50}$$

Here the maximum interval for θ equals 180° and for δ it equals 270°; as θ goes from $\pi/2$ through 0 to $-\pi/2$, δ goes from π through 0 to $-\pi/2$.

We compute now the *cross characteristics* for a *centered* wave. From (50), using polar coordinates r and δ we obtain (Fig. 6a)

$$\frac{r\,d\delta}{dr} = -\tan 2\bar{\varphi} = -\tan\left(\frac{\pi}{2} + \theta\right) = -\tan\frac{2\delta + \pi}{3} \tag{51}$$

$$r = r_0\left(\sin\frac{2\delta + \pi}{3}\right)^{-3/2}. \tag{52}$$

We finally compute for the same example the *principal stress lines*. For the ξ-lines, the lines that make the angle θ with the x-direction, we have, in polar coordinates (Fig. 6b)

$$\frac{r\,d\delta}{dr} = -\tan(\delta - \theta) = -\tan\left(\frac{\delta}{3} + \frac{\pi}{6}\right) \tag{53}$$

or

$$r = r_0\left[\sin\frac{2\delta + \pi}{6}\right]^{-3} \tag{54}$$

and for the η-lines

$$r = r_0\left[\cos\frac{2\delta + \pi}{6}\right]^{-3}. \tag{55}$$

These examples may suffice.

Simple waves are widely used in the solution of actual problems.

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–NOTES–

NOTE ON THE TORSIONAL RIGIDITY OF SEMI-CIRCULAR BARS*

By P. L. SHENG (*Taipeh, Taiwan*)

In such cases, the general stress function for torsion for the cross-section of a circular sector[1] reduces to

$$\Psi = \frac{1}{2}\left[-r^2(1 + \cos 2\theta) + \frac{16a^2}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} (-1)^{(n+1)/2}\left(\frac{r}{a}\right)^n \frac{\cos n\theta}{n(n^2 - 4)}\right],$$

where r, θ are the polar coordinates, and a the radius of the semi-circle. It is easy to verify that $\Psi = 0$ at the boundary.

The applied torque T is given by the well-known formula

$$T = 2G\alpha \int_A \Psi \, dA = 2G\alpha \iint_A \Psi r \, dr \, d\theta,$$

where $\int_A$ or $\iint_A$ are taken over the cross-sectional area, G denotes the shearing modulus of elasticity, and α the angle of twist per unit length of the bar.

Hence

$$\begin{aligned} T = G\alpha\Big[&- \int_{-\pi/2}^{\pi/2} \int_0^a r^3(1 + \cos 2\theta) \, dr \, d\theta \\ &+ \frac{16a^3}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{(-1)^{(n+1)/2}}{n(n^2 - 4)} \int_{-\pi/2}^{\pi/2} \int_0^a \left(\frac{r}{a}\right)^{n+1} \cos n\theta \, dr \, d\theta \Big] \\ = G\alpha\Big[&- \frac{\pi}{4} a^4 - \frac{32a^4}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{(n - 2)n^2(n + 2)^2} \Big]. \end{aligned}$$

Using the well-known formula

$$\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8},$$

we can easily derive by the split

$$\frac{1}{(n - 2)n^2(n + 2)^2} = \frac{1}{32}\left[\frac{1}{(n - 2)n} + \frac{5}{n(n + 2)} - \frac{4}{n^2} - \frac{2}{(n + 2)^2}\right]$$

that

$$\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{(n - 2)n^2(n + 2)^2} = \frac{1}{8}\left[1 - \frac{3}{16}\pi^2\right].$$

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*Received September 30, 1950.

[1]Cf., e.g., S. Timoshenko's *Theory of elasticity*, §79, noticing that in Timoshenko's book the stress function ϕ is $\phi = G\alpha\Psi$, and T, α, and (r, θ) in the present paper are written as M_t, θ, and (r, ψ) respectively.

By substitution,

$$T = G\alpha a^4\left[\frac{\pi}{2} - \frac{4}{\pi}\right]$$

exactly, and the numerical expression is

$$T = 0.297{,}556{,}782\ G\alpha a^4$$

which differ slightly from St. Venant's result of 0.296 or 0.2966.[2]

[2]The value 0.296 is from the table in S. Timoshenko's *Theory of elasticity*, p. 250, 1st ed., 2nd and 8th impression (1933 and 1934), McGraw-Hill; while from the datum given in I. Todhunter and K. Pearson's *A history of the theory of elasticity and of the strength of materials*, vol. II, part I, p. 193, we have

$$T = 0.3776M = 0.3776 \times G\alpha \times (\pi a^2)/2 \times a^2/2 = 0.2966G\alpha a^4.$$

A NOTE ON MY PAPER

ON STEADY LAMINAR TWO-DIMENSIONAL JETS IN COMPRESSIBLE VISCOUS GASES FAR BEHIND THE SLIT*

Quarterly of Applied Mathematics, 7, 313-323 (1949)

By M. Z. KRZYWOBLOCKI (*University of Illinois*)

Determination of the constant of integration for the temperature distribution (p. 317, eq. 25) from the condition that that total flux of enthalpy across jet is alike at all cross-sections restricts solution to small Mach numbers (if the comparison cross-section is close to the slit, as pointed out by A. H. Shapiro, Appl. Mech. Rev. III (1950) p. 415, No. 2718) or to high Mach numbers (if the comparison section is far from the slit). To take into account all the relative cases, that constant may be determined from the condition that the total flux (enthalpy plus kinetic energy) is alike at all cross-sections:

$$2\int_0^\infty (Jc_pT_1 + u_1^2/2)u_1(\rho_\infty + \rho_1)\,dy = \text{const.}$$

*Received Feb. 22, 1951.

ON THE LEAST EIGENVALUE OF HILL'S EQUATION*

By C. R. PUTNAM (*The Institute for Advanced Study, Princeton*)

The differential equation

$$x'' + [\lambda + f(t)]x = 0, \qquad (1)$$

in which λ is a real parameter and $f(t)$, for $-\infty < t < \infty$, is a real-valued, continuous, periodic function ($\not\equiv 0$), arises in problems dealing with the propagation of waves in

*Received November 10, 1950.

periodic media; cf., e.g., [1], [5], [6]. For a proper choice of units on the t-axis it may be supposed that $f(t)$ has period 1 and hence possesses a Fourier series

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}, \qquad (c_{-n} = \bar{c}_n). \tag{2}$$

There exists a sequence of finite intervals (the intervals of stability) $I_k : \lambda_k \leqq \lambda \leqq \lambda^k$, where $\lambda_k < \lambda^k < \lambda_{k+1}$ and $k = 1, 2, \cdots$, such that (1) possesses, or fails to possess, a solution x ($\not\equiv 0$) which is bounded on $-\infty < t < \infty$ according as λ does, or does not, belong to the (closed) set $S = \sum I_k$; cf. [6], p. 16. It is known ([8], [3]) that the set S is identical with the invariant spectrum (Weyl [7], p. 251) associated with the differential equation (1), on either the full line $-\infty < t < \infty$ or a half-line, say, $0 \leqq t < \infty$.

Let λ_1 be denoted by μ so that μ is the least point of the set S. Alternatively, μ can be defined by the requirement that (1) be oscillatory, that is, every solution of (1) should possess an infinity of zeros on $0 \leqq t < \infty$ (or equivalently, in the present case, on $-\infty < t < \infty$) whenever $\lambda > \mu$, and non-oscillatory whenever $\lambda < \mu$. (This twofold characterization of μ holds also in the general case of (1) in which f need not be periodic; cf. [2].)

Wintner ([9], p. 116, and [10]) has obtained the following estimates of μ in terms of the Fourier coefficients of the function $f(t)$ defined by (2):

$$-c_0 - 2\sum_{n=1}^{\infty} | c_n |^2 \leqq \mu \leqq -c_0 . \tag{3}$$

The present note will be devoted to the problem of obtaining other such estimates. A formula for μ involving the Fourier coefficients of the given function $f(t)$ and those of arbitrary periodic functions, subject to certain specified conditions, will be deduced (cf. (12) below); furthermore, as a corollary of this formula, upper bounds for μ, involving not only c_0, as in (3), but also arbitrary coefficients c_n, will be obtained. Specifically, it will be shown that

$$\mu \leqq \pi^2 N^2 - c_0 + \Re(c_N), \qquad N = 1, 2, \cdots, \tag{4}$$

where $\Re(c)$ denotes the real part of a complex number c. In case $f(t)$ satisfies

$$-f(t) = f(t + c),$$

for some real number c, as, e.g., is the case if $f = \sin 2\pi t$, it is clear from the properties of μ that (4) can be refined to

$$\mu \leqq \pi^2 N^2 - c_0 - | \Re(c_N) |, \qquad N = 1, 2, \cdots .$$

2. It follows from [4] (cf. the Remark on p. 636) that μ satisfies

$$\mu = \lim_{T\to\infty} \left\{ \text{g.l.b.} \left[\int_T^{\infty} (x'^2 - fx^2)\, dt \Big/ \int_T^{\infty} x^2\, dt \right] \right\}, \tag{5}$$

where $x(t)$ belongs to the class of functions, Ω_T, which, on the half-line $T \leqq t < \infty$, are real-valued, continuous, and have piecewise continuous first derivatives (with respect to any finite subinterval of $T \leqq t < \infty$) and, furthermore, satisfy

$$x(T) = 0, \qquad 0 < \int_T^{\infty} x^2\, dt < \infty, \qquad \int_T^{\infty} x'^2\, dt < \infty. \tag{6}$$

Since $f(t)$ is periodic, it is clear that the expression $\{\cdots\}$ occurring in (5) is independent of T. Hence (5) remains valid if the limit sign is removed and the expression $[\cdots]$ of (5) is evaluated only for functions of class Ω_0.

Let $x(t)$ denote any function of class Ω_0. Clearly, it is possible to define a function $y(t)$ in Ω_0, such that $y(t) \equiv 0$ for sufficiently large t, and, in addition, is such that the expression $[\cdots]$ of (5) (for $T = 0$) evaluated for y differs from the corresponding expression for x by less than an arbitrarily preassigned positive number. (In fact, since $\int_0^\infty x^2\,dt < \infty$, there exists a sequence of points t_n such that $t_n \to \infty$ and $x(t_n) \to 0$, as $n \to \infty$. Let $z_n(t)$, where $n = 1, 2, \cdots$ and $t_n \leqq t \leqq t_n + 1$, be a sequence of continuous functions, with piecewise continuous first derivatives, such that $z_n(t_n) = x(t_n)$, $z_n(t_n + 1) = 0$, and

$$\int_{t_n}^{t_n+1} z^2\,dt \to 0, \qquad \int_{t_n}^{t_n+1} z_n'^2\,dt \to \infty, \qquad \text{as } t_n \to \infty.$$

If $y_n = x$ on $0 \leqq t \leqq t_n$, $y_n = z_n$ on $t_n \leqq t \leqq t_n + 1$ and $y_n \equiv 0$ for $t_n + 1 \leqq t < \infty$, it is clear that y_n satisfies the conditions claimed for the function y above, provided t_n is sufficiently large.) The corners of y can be "smoothed out" so that, in addition to the properties required above, y has a continuous first derivative on $0 \leqq t < \infty$.

It follows from the above discussion that μ can be defined by

$$\text{g.l.b.}\left[\int_0^Q (x'^2 - fx^2)\,dt \Big/ \int_0^Q x^2\,dt\right], \tag{7}$$

where x belongs to Γ_Q, the set of (real-valued) functions $x(t)$ ($\not\equiv 0$) which possess continuous first derivatives on $0 \leqq t \leqq Q$ and satisfy $x(0) = x(Q) = 0$, and Q is, for convenience, an arbitrary (variable) positive integer. Any function x of Γ_Q, and its derivative x', can be uniformly approximated on $0 \leqq t \leqq Q$ by a sequence of trigonometric polynomials of Γ_Q, and their derivatives, respectively. Hence, μ can be defined by (7) where, now, $x(t)$ is any function of the type

$$x(t) = \sum_{n=-N}^{N} a_n e^{2\pi i n (P/Q) t}, \qquad a_{-n} = \bar{a}_n, \qquad \sum_{n=-N}^{N} a_n = 0, \tag{8}$$

and where N, P and Q denote arbitrary positive integers. The remainder of this section will be devoted to obtaining a transcription of (7) in terms of the Fourier coefficients of the given f, defined by (2), and the Fourier coefficients of the variable function x, defined by (8).

Let P and Q denote relatively prime positive integers and note that (2) and the first relation of (8) can be rewritten as

$$f(t) \sim \sum C_n e^{2\pi i n t/Q} \qquad \text{and} \qquad x(t) = \sum A_n e^{2\pi i n t/Q}, \tag{9}$$

where

$$c_n = C_{Qn} \qquad \text{and} \qquad a_n = A_{Pn} \qquad \text{for } n = 0, \pm 1, \cdots, \pm N. \tag{10}$$

(By the uniqueness theorem for Fourier series, $C_n = 0$ if Q does not divide n and $A_n = 0$ if P does not divide n.) Since $x^2 = \sum B_n e^{2\pi i n t/Q}$, where $B_n = \sum_k A_k A_{n-k}$, it follows from the Parseval relation that

$$\int_0^Q x^2\,dt = Q \sum |A_n|^2, \qquad \int_0^Q x'^2\,dt = (4\pi^2/Q) \sum n^2 |A_n|^2$$

and

$$\int_0^Q fx^2\,dt = Q \sum \bar{C}_n B_n .$$

Thus, the expression $[\cdots]$ of (7) is equal to

$$\left[(4\pi^2/Q^2) \sum n^2 \mid A_n \mid^2 - Q \sum_n \bar{C}_n\Big(\sum_k A_k A_{n-k}\Big)\right]\Big/ Q \sum \mid A_n \mid^2. \tag{11}$$

From the definition (10) of C_n and A_n, it is clear that, in the numerator of (11), the n occurring in the first term may be replaced by Pn, while the n and k of the second term may be replaced by Qn and Pk respectively, so that the summation occurring in the second term of the numerator of (11) becomes $\sum_n \bar{C}_{Qn} \left(\sum_k A_{Pk}A_{Qn-Pk}\right)$. Since P and Q are relatively prime, $\sum_k A_{Pk}A_{Qn-Pk} = 0$ unless n is a multiple of P. But $A_{QPn-Pk} = A_{P(Qn-k)}$; hence, from (10), (11) and (7), there follows

$$\mu = \text{g.l.b.}\left\{\left[(4\pi^2P^2/Q^2) \sum n^2 \mid a_n \mid^2 - \sum_n \bar{c}_{Pn}\Big(\sum_k a_k a_{Qn-k}\Big)\right]\Big/ \sum \mid a_n \mid^2\right\}. \tag{12}$$

In the expression (12), the arbitrary positive integers P and Q and the finite set of complex numbers $a_{-N}, a_{-N+1}, \cdots, a_N$ are subject to

$$(P, Q) = 1, \qquad a_{-n} = \bar{a}_n \qquad \text{and} \qquad \sum_{n=-N}^{N} a_n = 0. \tag{13}$$

(It is to be noticed that each summation of (12) extends over only a finite range.) The formula (12), subject to (13), will be used in the next section to deduce (4).

3. Choose $a_1 = 1/2i$, $a_{-1} = -1/2i$, $a_n = 0$ if $n \neq \pm 1$; let $Q = 1$ and let P denote an arbitrary positive integer. (This selection corresponds to the choice $x = \sin 2\pi Pt$ in (8).) The expression $\{\cdots\}$ of (12) becomes $\pi^2(2P)^2 - c_0 + \Re(c_{2P})$ so that the equation (4) is proved for an arbitrary even positive integer $N = 2P$. Suppose now that N of (4) is odd and choose $P = N$ and $Q = 2$; let the a_n's be defined as before. Proceeding as above one sees that relation (12) implies $\mu \leqq \pi^2N^2 - c_0 + \Re(c_N)$ and the proof of (4) is complete.

Appendix

I. For computational purposes, one may easily verify that the formula (12) can be modified to

$$\mu = -c_0 + \text{g.l.b.}\left\{\left[(8\pi^2P^2/Q^2) \sum_{n=1}^{N} n^2 \mid a_n \mid^2 \right.\right.$$
$$\left.\left. - 2\Re\Big(\sum_{n=1}^{N} \bar{c}_{Pn}\Big(\sum_k a_k a_{Qn-k}\Big)\Big)\right]\Big/\Big(a_0^2 + 2 \sum_{n=1}^{N} \mid a_n \mid^2\Big)\right\}.$$

For appropriate choices of the sequences $\{a_i\}$ and the pairs of integers P and Q, subject to (13), various upper bounds for μ, in addition to those of (4), may be readily obtained from the above formula and (13). Non-trivial lower bounds for μ are not as easily obtained, in this manner, corresponding to the presence of the symbol "g.l.b." in the formulas for μ.

II. The question has been raised by Wintner [10] whether the constant 2 occurring as the coefficient in (3) is the least value of α for which

$$\mu \geq -c_0 - \alpha \sum_{n=1}^{\infty} | c_n |^2 \tag{14}$$

holds for an arbitrary periodic function $f(t)$ defined by (2). Although this question will remain unanswered, it can easily be shown, as a consequence of (4), that $\alpha \geqq 1/4\pi^2$. For, suppose (14) holds for all $f(t)$ defined by (2); then, by (4), $-\alpha \sum_{n=1}^{\infty} | c_n |^2 \leqq \pi^2 N^2 + \Re(c_N)$ holds for $N = 1, 2, \cdots$. If c_N is real, it follows that $\pi^2 N^2 + c_N + \alpha c_N^2 \geqq 0$; hence, by a consideration of the discriminant of this last quadratic expression, $1 - 4\alpha\pi^2 N^2 \leqq 0$. For $N = 1$, this implies $\alpha \geqq 1/4\pi^2$, which was to be shown.

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A MODIFICATION OF SOUTHWELL'S METHOD*

By W. H. INGRAM (*New York*)

J. L. Synge[1] has given a geometrical interpretation of Southwell's method of solution of the problem $Ax = b$ when $A = (a_{ij})$ is symmetric and $\sum \sum a_{ij}x_ix_j$ is a positive definite form. A modification of the method having application to the more general case in which xA_TAx is a positive definite form makes use of the ellipsoids of the Gauss-Seidel process.

For any vector x, there is an error e defined by the equation

$$Ax - b = e, \tag{1}$$

therefore

$$(xA_T - b)W(Ax - b) = eWe; \tag{2}$$

*Received Jan. 2, 1951.

[1]J. L. Synge, *A geometrical interpretation of the relaxation method*, Q. Appl. Math., **2**, p. 87 (1944).

W is a diagonal weighting matrix for weighting the relative importance of the equations of the set (1). An orthogonal transformation T exists such that

$$T_T A_T W A T = D \equiv [d_1, d_2, \cdots, d_n)$$

is a diagonal matrix of positive elements d_i. The substitution

$$x = Ty, \qquad x = yT_T,$$

into (2) gives

$$d_1 y_1^2 + d_2 y_2^2 + d_3 y_3^2 + \cdots - 2\beta_1 y_1 - 2\beta_2 y_2 - 2\beta_3 y_3 - \cdots = eWe - bWb, \tag{3}$$

an equation in which $\beta = T_T A_T W b$. It is seen that (3) is the equation of a family of hyperellipsoids with the properties

(a) a common center at $(\beta_1/d_1, \beta_2/d_2, \cdots)$
(b) a common orientation,
(c) common principal axes' ratios,
(d) parameter e.

To complete the squares, one adds $\sum \beta_i^2/d_i$ to both sides of (3):

$$\sum \beta_i^2/d_i = \beta D^{-1}\beta = bATD^{-1}T_T A_T b = bWb.$$

It is seen that the ellipsoids converge to a point as $e \to 0$ and that their common center, given by $e = 0$, is the solution of (1).

The solution may be approximated in the same way as for the family of ellipsoids employed by Southwell and by the same process.

THE LAPLACIAN AND MEAN VALUES*

By R. M. REDHEFFER and R. STEINBERG (*University of California, Los Angeles*)

Introduction. The Laplacian $\nabla^2 f$ represents *deviation from the average* in the following sense [1]. Let f_0 be the mean value of f over a cube of edge $2t$,

$$8t^3 f_0 = \iiint_{-t}^{t} f(x + u, y + v, z + w)\, du\, dv\, dw. \tag{1}$$

Then

$$\lim_{t\to 0} 6(f_0 - f)/t^2 = \nabla^2 f, \tag{2}$$

provided f is sufficiently smooth. This result is interesting in that it is independent of the co-ordinate system. Also it sheds a certain light on the wave equation. Thus, if the restoring force at a point in a medium is proportional to the deviation from the average, in some sense, then one might expect the equation of motion to be $a\, \nabla^2 f = bf_{tt}$, where a measures stiffness, b inertia.

*Received April 9, 1951.

A harmonic function could be characterized as one such that the limit (2) is zero, in other words, such that its value at a point is close to its average over a suitable neighborhood. This seems interesting too, since a harmonic function can be characterized by an exact equality of that sort over the surface of a sphere.

A result of the type (2) is true in one or two dimensions, and also when the average is taken over the surface, edges or vertices of the cube, rather than over its volume. From (2) for the boundary of a square in two dimensions one can deduce the fact that the line integral of a harmonic function in two dimensions is independent of path. Again, (2) over the vertices of a square or cube is related to methods of numerical integration [2], where one wishes to find f_0 by computing f or its derivatives at a point.

The Problem. What we want to do here is to characterize the point sets for which a result like (2) holds. It is seen from [1] that symmetry in the coordinate planes through (x, y, z) is sufficient, but it is not necessary. For a solid set, f is taken as a triple integral, divided by the volume of the set; for an area one uses a double integral divided by the area; for a space curve one uses a line integral divided by the length; and for a finite point set one takes the sum of the values, divided by the number of the points. Conciseness requires that the result be given here in a form that applies to all these cases at once, and to any number of dimensions. Accordingly we denote the number, length, surface, or volume by a general measure m:

I. Let S be a bounded point set on which is defined a measure m such that $0 < m(S) = M < \infty$. If the vector P represents a fixed point of S, and the vector Q a variable point of S, the following statements are equivalent:

(a) The set S has mass M, its center of mass is at P, and its moment of inertia about every axis through P is I.

(b) $\int f[P + t(Q - P)]\, dm(Q) - Mf(P) \sim (t/2)^2 I\, \nabla^2 f(P)$ as $t \to 0$ for an arbitrary function f having continuous third derivatives near P.

We do not dwell on the proof, which is rather trivial. There is no restriction in taking $P = 0$. Then, Taylor's theorem with remainder about $P = 0$ (in Cartesian co-ordinates with $Q = (x, y, z)$) shows (b) to be equivalent to the mass and center of mass results of (a) and

$$\int x^2 = \int y^2 = \int z^2 = I/2, \tag{3}$$

$$\int yz = \int zx = \int xz = 0. \tag{4}$$

(In each integration Q ranges over S.)

The identity $x^2 = [(z^2 + x^2) + (x^2 + y^2) - (y^2 + z^2)]/2$ shows that (3) are equivalent to

$$\int (y^2 + z^2) = \int (z^2 + x^2) = \int (x^2 + y^2) = I. \tag{5}$$

Equations (4) and (5) are equivalent to the identity in l, m, n:

$$\int [(mz - ny)^2 + (nx - lz)^2 + (ly - mx)^2] \equiv (l^2 + m^2 + n^2)I;$$

and this is the moment of inertia condition of (a).

As for the speed with which the limit in (b) is approached, we get, again using Taylor's expansion, the following:

II. In I(b), the error is $O(t^3)$. If f has continuous fourth derivatives near P, the error can be made $O(t^4)$ by a further restriction on S. Whatever set S is chosen, the error cannot be made $o(t^4)$ even if the class of functions is restricted to polynomials.

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A NOTE ON ASYMPTOTIC STABILITY*

By H. A. ANTOSIEWICZ (*Montana State College*)

1. In this note we shall develop a stability criterion for a vector differential equation of the form

$$\frac{dx}{dt} = A(t)x, \tag{1}$$

where the elements of the matrix $A(t) = (a_{ij}(t))$, $i, j = 1, 2, \cdots n$, are real continuous and uniformly bounded functions for all positive $t \geqq t_0$.

A. Wintner** recently established the following criterion: Let $\lambda_1(t)$ be the greatest, and $\lambda_2(t)$ the least characteristic value of the matrix $\frac{1}{2}[A(t) + A'(t)]$, and let $\| x(t) \|$ denote the Euclidean length of the vector $x(t)$. If $\int^\infty \lambda_1(t)\, dt < \infty$, $\int^\infty \lambda_2(t)\, dt < \infty$, then $\| x(t) \| \to \kappa \neq 0$ as $t \to \infty$ for every non-trivial solution $x(t)$ of (1).

It is to be noted that the condition of integrability of $\lambda_1(t)$, $\lambda_2(t)$ over (t_0, ∞) implies $\int^\infty [\text{trace } A(t)]\, dt < \infty$. Furthermore, this condition automatically excludes the important case $A(t) =$ const. unless $A(t) =$ const. is skew-symmetric.

In the following we shall establish a stability criterion which is free of the above objection, i.e. which will also apply to the general case $A(t) =$ const. We shall consider a condition to be satisfied by the matrix $A(t)$ which will suffice to insure that $\| x(t) \|$ of every non-trivial solution $x(t)$ of (1) tends to zero as $t \to \infty$. According to Liapounoff†, the trivial solution $x(t) \equiv 0$ is then said to be asymptotically stable.

2. Consider a function $V(x, t)$ which is defined and continuous for all x and t in R: $| x_i | \leqq c$, $t \geqq T$ $(i = 1, 2, \cdots n)$. If for equation (1) there exists in R a function $V(x, t)$ which is of fixed sign and admits of an infinitely small upper bound, and for which dV/dt by virtue of (1) is opposite in sign to $V(x, t)$ in R, then the trivial solution $x(t) \equiv 0$ of (1) is asymptotically stable. Liapounoff proved that the existence of such a function

*Received April 16, 1951.

A. Wintner, *On free vibrations with amplitudinal limits*, Quart. Applied Math. **8, 102-10: (1950).

†A. Liapounoff, *Problème général de la stabilité du mouvement*, Ann. Math. Studies, No. 17, 1949.

$V(x, t)$ is sufficient for asymptotic stability; it is, however, not necessary as was shown by J. Malkin.*

We shall make use of Malkin's results to establish the following theorem:

Let $\lambda_1(t)$ be the greatest, and $\lambda_2(t)$ the least characteristic value of the matrix $\frac{1}{2}[A(t) + A'(t)]$. If $\int^t \lambda_1(\tau)\, d\tau \to -\infty$, $\int^t \lambda_2(\tau)\, d\tau \to -\infty$ as $t \to \infty$, then $\| x(t) \| \to 0$ as $t \to \infty$ for every non-trivial solution $x(t)$ of (1), i.e. the trivial solution $x(t) \equiv 0$ is asymptotically stable.

Note that now $\int^\infty$ [trace $A(t)]dt$ diverges.

3. First, we transform (1) into diagonal form. Let $x_1, x_2, \cdots x_n$ be a base of solutions of (1), and use this base to construct an orthogonal matrix $C(t)$. If $y = C^{-1}(t)x$, then (1) reduces to

$$\frac{dy}{dt} = B(t)y, \qquad B(t) = C^{-1}AC + \frac{dC^{-1}}{dt}C \tag{2}$$

where the matrix $B(t) = (b_{ij}(t))$, $i, j = 1, 2, \cdots n$, is diagonal, i.e. $b_{ij}(t) \equiv 0$ for all $i > j$. If $y_1(t), y_2(t), \cdots y_n(t)$ is that base of solutions of (2) for which $y_i(t_0) = I^i$, the i-th column vector of the identity matrix I, then $\| x_i(t) \| = \| y_i(t) \|$ as is easily verified. Evidently, $C(t)$ and $C^{-1}(t)$ have bounded elements and $| C(t) | = | C^{-1}(t) | = 1$; hence stability properties are preserved in both directions.

Observing that $C^{-1}(t) = C'(t)$ by construction, we find by differentiating the identity $C(t)C^{-1}(t) \equiv I$ that $(dC^{-1}/dt)C$ is skew-symmetric. Therefore $B(t) + B'(t) = C^{-1}[A(t) + A'(t)]C$, and thus the characteristic values of $\frac{1}{2}[B(t) + B'(t)]$ are identical with those of $\frac{1}{2}[A(t) + A'(t)]$. Hence it is sufficient to prove our theorem for the reduced equation (2). We shall show that there exists a function $V(y, t)$ which satisfies Liapounoff's criterion for asymptotic stability.

Consider the diagonal elements $b_{ii}(t)$, $i = 1, 2, \cdots n$, of the matrix $B(t)$. Since $(dC^{-1}/dt)C$ is skew-symmetric, trace $(dC^{-1}/dt)C \equiv 0$, and thus

$$b_{ii}(t) = (C^{-1})_i AC^i = (C^i)'AC^i = \tfrac{1}{2}(C^i)'[A(t) + A'(t)]C^i. \tag{3}$$

All diagonal elements of $B(t)$ are quadratic forms in the components of the column vectors C^i of the matrix $C(t)$ for which we evidently have $\| C^i \| = 1$. These quadratic forms attain their maximum and minimum on the unit sphere $\| C^i \| = 1$ (compact set); if $\lambda_1(t)$ is the greatest, $\lambda_2(t)$ the least characteristic value of $\frac{1}{2}[B(t) + B'(t)]$, then $\lambda_1(t)$ is the maximum, $\lambda_2(t)$ the minimum. From (3) we then obtain

$$\lambda_1(t) \geqq b_{ii}(t) \geqq \lambda_2(t) \tag{4}$$

whence for all $t \geqq t_0$

$$\exp\left(\int_{t_0}^t \lambda_1(\tau)\, d\tau\right) \geqq \exp\left(\int_{t_0}^t b_{ii}(\tau)\, d\tau\right) \geqq \exp\left(\int_{t_0}^t \lambda_2(\tau)\, d\tau\right). \tag{5}$$

By hypothesis $\int^t \lambda_k(\tau)\, d\tau \to -\infty$ as $t \to \infty$, $k = 1, 2$, and thus

$$\varphi_i(t) = \exp\left(\int_{t_0}^t b_{ii}(\tau)\, d\tau\right) \to 0 \qquad \text{as} \qquad t \to \infty. \tag{6}$$

As Malkin has shown, (6) involves for all $t \geqq t_0$

$$\varphi_i(t) \int_{t_0}^t \frac{d\tau}{\varphi_i(\tau)} \leqq c \tag{7}$$

*J. Malkin, *Certain questions on the theory of the stability of motion in the sense of Liapounoff*, American Math. Soc., Translation No. 20, 1950.

and (6) and (7) together, in turn, imply $\int^\infty [\varphi_i(t)]^2\, dt < \infty$. Hence the functions

$$\psi_i(t) = [\varphi_i(t)]^{-2} \int_t^\infty [\varphi_i(\tau)]^2\, d\tau \tag{8}$$

exist for all $t \geqq t_0$ and are uniformly bounded; in fact, $a^2 \leqq \psi_i(t) \leqq b^2$ where a and b are certain constants.

Now consider the function

$$V(y, t) = \psi_1(t)y_1^2 + \psi_2(t)y_2^2 + \cdots + \psi_n(t)y_n^2 .$$

It evidently satisfies Liapounoff's criterion for asymptotic stability; it is a positive definite quadratic form, admitting of an infinitely small upper bound, and its derivative, by virtue of (2), becomes

$$\frac{dV}{dt} = -(y_1^2 + y_2^2 + \cdots + y_n^2) + W(y, t)$$

where $W(y, t)$ is a quadratic form whose coefficients depend upon those elements $b_{ij}(t)$ of $B(t)$ for which $i < j$, $i, j = 1, 2, \cdots n$. Since these elements can always be made sufficiently small by a transformation with constant coefficients (which will not affect stability properties) the derivative dV/dt will be a negative definite quadratic form. Hence the trivial solution $y(t) \equiv 0$ of (2) is asymptotically stable, and therefore the trivial solution $x(t) \equiv 0$ of (1) is asymptotically stable. This establishes our theorem.

O. Perron* was the first to prove directly that the conditions

$$\varphi_i(t) \leqq C_1 , \qquad \varphi_i(t) \int_{t_0}^t \frac{d\tau}{\varphi_i(\tau)} \leqq C_2$$

are necessary and sufficient for the trivial solution $x(t) \equiv 0$ of (1) to be asymptotically stable.

*O. Perron, *Die Stabilitaetsfrage bei Differentialgleichungen*, Math. Zeitschrift **32**, 703-728 (1930).

CONDITIONS SATISFIED BY THE EXPANSION AND VORTICITY OF A VISCOUS FLUID IN A FIXED CONTAINER*

By J. L. SYNGE (*Dublin Institute for Advanced Studies*)

1. Introduction. In plane motion of a viscous fluid inside a fixed container, the expansion θ and the vorticity ω cannot be arbitrarily assigned. A necessary and sufficient condition[1] for the consistency of given θ and ω with vanishing velocity on the walls is

$$\int (\theta U + 2\omega V)\, dS = 0, \tag{1.1}$$

*Received April 17, 1951. This paper was written while the author was on leave of absence at the Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.

[1]J. L. Synge, Quarterly of Applied Mathematics, **8**, 107-108 (1950). The condition with $\theta = 0$ was originally due to G. Hamel, Göttinger Nachr. Math.-Phys. Kl. **1911**, 261-270.

where U, V is any pair of conjugate harmonic functions and the integration is taken over the region occupied by the fluid.

The purpose of the present paper is to extend this result to three dimensions. In space we have expansion θ and a vorticity vector ω_i (suffixes range 1, 2, 3 with the summation convention), and the theorem which will be proved may be stated as follows:

Given expansion θ and vorticity ω_i are consistent with vanishing velocity on the walls if, and only if,

$$\int (\theta P + \omega_i Q_i)\, dV = 0, \tag{1.2}$$

where the integration is taken through the fluid, Q_i being any (arbitrary) solution of the partial differential equations

$$\Delta Q_i = Q_{k,ki} \tag{1.3}$$

and P satisfying

$$P_{,i} = \tfrac{1}{2}\epsilon_{ijk} Q_{k,j} \, . \tag{1.4}$$

Here ϵ_{ijk} is the usual permutation symbol, the comma denotes partial differentiation ($Y_{,i} = \partial Y/\partial x_i$), and Δ the Laplace operator, so that $\Delta Q_i = Q_{i,kk}$. In vector notation (1.3) and (1.4) read $\nabla^2 \mathbf{Q} = \nabla\nabla\cdot\mathbf{Q}$ and $\nabla P = \frac{1}{2}\nabla \times \mathbf{Q}$.

This theorem will be considered only for simply connected regions. For such regions, (1.3) are precisely the integrability conditions of (1.4), so that, given any solution of (1.3), P exists satisfying (1.4), unique to within an additive constant.

2. Necessity of condition (1.2). Expansion θ and vorticity ω_i are connected with velocity u_i by

$$u_{i,i} = 0, \qquad \epsilon_{ijk} u_{k,j} = 2\omega_i \, . \tag{2.1}$$

The problem of finding a motion with given θ and ω_i in a region V, bounded by a fixed surface B to which the fluid adheres, is the problem of solving the partial differential equations (2.1) for u_i with the boundary condition

$$u_i = 0 \qquad \text{on } B. \tag{2.2}$$

The theorem stated above asserts that (1.2) is a necessary and sufficient condition on θ and ω_i for the existence of this solution.

The necessity of the condition is easy to prove. We assume the existence of a solution of (2.1) and (2.2).

By virtue of (2.1), it follows that for any P and Q_i at all (not subject to any conditions save those of smoothness) we have

$$\int (\theta P + \omega_i Q_i)\, dV + \int u_i(P_{,i} - \tfrac{1}{2}\epsilon_{ijk} Q_{k,j})\, dV$$

$$= \int [(u_i P)_{,i} + \omega_i Q_i - (\tfrac{1}{2}\epsilon_{ijk} u_i Q_k)_{,j} + \tfrac{1}{2}\, \epsilon_{ijk} u_{i,j} Q_k]\, dV \tag{2.3}$$

$$= \int (n_i u_i P - \tfrac{1}{2}\, n_j \epsilon_{ijk} u_i Q_k)\, dB,$$

the last integral being taken over the boundary B, on which n_i is the unit normal, drawn outward.

Note that the above follows from (2.1) only. We now bring in (2.2). This makes the last integral in (2.3) vanish. If we then subject P and Q_i to (1.3) and (1.4), the second integral in the first line vanishes, and we are left with the equation (1.2); the *necessity* of (1.2) is thus established.

3. A lemma. The proof of sufficiency is harder. It rests on a lemma, for which the proof offered here is not mathematically rigorous, resting as it does on the assumption that a certain minimum is attained. A precise mathematical proof would of course have to specify the requisite smoothness of the bounding surface B and of the tangential component assigned on it (see immediately below).

Lemma: Given the tangential component of a vector Q_i on the boundary B of a region V, then Q_i exists satisfying (1.3) *and this boundary condition.*

To prove this (or at least make it plausible), consider the integral

$$I(Q) = \int \epsilon_{ijk} Q_{k,j} \epsilon_{irs} Q_{s,r} \, dV. \tag{3.1}$$

In vector notation, the integrand is $(\nabla \times \mathbf{Q})^2$ and cannot be negative. Thus for all Q_i satisfying the stated boundary condition, $I(Q)$ is bounded below. We assume that the minimum is actually attained by some vector field; let Q_i be it.

Then, if c is any constant, and q_i any vector field with zero tangential component on B, it follows that

$$I(Q) \le I(Q + cq), \tag{3.2}$$

and hence by the usual procedure associated with Dirichlet's principle,

$$\int \epsilon_{ijk} Q_{k,j} \epsilon_{irs} q_{s,r} \, dV = 0 \tag{3.3}$$

for all such q_i. This may be transformed into

$$\int \epsilon_{ijk} Q_{k,j} \epsilon_{irs} q_s n_r \, dB - \int \epsilon_{ijk} Q_{k,jr} \epsilon_{irs} q_s \, dV = 0. \tag{3.4}$$

But $\epsilon_{irs} q_s n_r$ is the tangential component of q_i, turned through a right angle, and so vanishes. Further

$$\epsilon_{ijk} \epsilon_{irs} = \delta_{jr} \delta_{ks} - \delta_{js} \delta_{kr}, \tag{3.5}$$

and so we get

$$\int (\Delta Q_k - Q_{i,ik}) q_k \, dV = 0. \tag{3.6}$$

Since q_i is arbitrary except for the boundary condition it follows that the minimising Q_i satisfies (1.3).

As pointed out already, P then exists satisfying (1.4).

4. The sufficiency of (1.2). We assume that θ and ω_i satisfy (1.2) for all Q_i and P satisfying (1.3) and (1.4). We have to prove the existence of a solution u_i of (2.1) and (2.2).

Choose a particular solution of (1.3), (1.4): $P = 1$, $Q_i = 0$. Then (1.2) gives

$$\int \theta \, dV = 0. \tag{4.1}$$

Now choose $P = 0$, $Q_i = W_{,i}$, the gradient of any scalar field W; these satisfy (1.3), (1.4) without restriction on W. Then (1.2) gives

$$\int \omega_i W_{,i} \, dV = 0, \tag{4.2}$$

or

$$\int \omega_i n_i W \, dB - \int \omega_{i,i} W \, dV = 0. \tag{4.3}$$

Hence, in view of the arbitrariness of W,

$$\omega_{i,i} = 0 \quad \text{in } V, \qquad \omega_i n_i = 0 \quad \text{on } B. \tag{4.4}$$

We now try to solve (2.1) and (2.2). This we do in two steps, first solving the system

$$u_{i,i} = \theta, \qquad \epsilon_{ijk} u_{k,j} = 2\omega_i , \tag{4.5}$$

with the boundary condition

$$u_i n_i = 0 \quad \text{on } B. \tag{4.6}$$

There is a well known procedure for this; we express u_i in terms of a scalar potential and a vector potential, and obtain for these certain Poisson integrals, yielding particular solutions of (4.5). The problem of solving (4.5) and (4.6) is thus reduced to a Neumann problem, and it is known that (4.1) and (4.4) are sufficient conditions for the existence of a solution to (4.5) and (4.6).

The second step is to prove that the *tangential* component of this u_i vanishes, the normal component being already zero by (4.6). This is easy. Equation (2.3) is valid, since it depends only on (2.1), i.e. (4.5). But the first two integrals in (2.3) vanish by (1.2) and (1.4) respectively. Also the first part of the last integrand vanishes by (4.6). So we are left with

$$\int n_j \epsilon_{ijk} u_i Q_k \, dB = 0. \tag{4.7}$$

But $\epsilon_{ijk} n_j Q_k$ is the tangential component of Q_i, turned through a right angle, and this, as we saw in the lemma, may be chosen arbitrarily. From this it follows at once that the tangential component of u_i must vanish on B.

Thus the solution of (4.5) and (4.6) is in fact the solution of (2.1) and (2.2). This completes the proof of the *sufficiency* of the condition (1.2), subject of course to the assumption that the minimum of (3.1) is in fact attained.*

Added in proof, June 22, 1951:* In recent papers which the author had not seen when the present paper was written, C. Truesdell, Comptes Rendus Ac. Sci. Paris, **232, 1277, 1396 (1951), has given the condition (1.2) with consideration of its sufficiency, and this condition has also been given by F. H. van den Dungen, Q. Appl. Math. **9**, 203 (1951), but the question of sufficiency has not been fully considered by him.

CONFORMAL MAPPINGS FOR CERTAIN DOUBLY CONNECTED DOMAINS*

By J. A. McFADDEN (*University of Michigan*)**

1. Introduction. In this paper conformal mappings are derived for some doubly connected domains, the inner boundary being formed by slots and the outer boundary being the unit circle. All these mapping functions utilize elliptic functions; those for the more complicated domains involve elliptic functions of two different moduli. The domains appear, for example, in the linearized theory of supersonic flow past conical bodies.

2. Mapping of domains bounded by a slot and the unit circle. Consider first an annular domain in the ζ_1-plane, $r_1 < |\zeta_1| < 1$, and the corresponding rectangular domain in the z_1-plane, $-2K_1 < x_1 < 2K_1$, $0 < y_1 < K_1'$, obtained by a logarithmic transformation from the ζ_1-plane. (See Fig. 1. Circular domains will be designated by Greek letters, $\zeta_i = \xi_i + i\eta_i$, and rectangular domains by Roman letters, $z_i = x_i + iy_i$.)

$$z_1 = (2K_1/\pi i) \log \zeta_1 \qquad (0 < \arg \zeta_1 < 2\pi) \tag{1}$$

The radius r_1 of the inner circle in the ζ_1-plane is given by the expression

$$r_1 = \exp(-\pi K_1'/2K_1). \tag{2}$$

The circle $|\zeta_1| = r_1$ can be collapsed onto a horizontal slot, $-(1 - k_1')/k_1 \leq \xi' \leq (1 - k_1')/k_1$, $\eta' = 0$, (see Fig. 1), with the circle $|\zeta_1| = 1$ mapping onto the circle $|\zeta'| = 1$, by means of the transformation

$$\zeta' = \frac{\operatorname{cn}(z_1 ; k_1) + ik_1' \operatorname{sn}(z_1 ; k_1)}{\operatorname{dn}(z_1 ; k_1)}, \tag{3}$$

where $\operatorname{sn}(z_1 ; k_1)$, $\operatorname{cn}(z_1 ; k_1)$, and $\operatorname{dn}(z_1 ; k_1)$ are the Jacobian elliptic functions with argument z_1 and modulus k_1, and where $k_1' = (1 - k_1^2)^{1/2}$. (See, for example, reference [1].) The constants K_1 and K_1' in equation (2) may then be identified as the complete elliptic integrals of the first kind having moduli k_1 and k_1', respectively. An alternative form of equation (3) is the following:

$$\begin{gathered} -(1 + \zeta'^2)/2\zeta' = \operatorname{sn}(z_1 - K_1 ; k_1) \\ (-2K_1 < x_1 < 2K_1 ; \qquad 0 < y_1 < K_1') \end{gathered} \tag{3a}$$

In this form we can understand the reason for the appearance of the Jacobian elliptic functions. The left member of (3a) maps the two boundaries in the ζ'-plane onto the real axis of an intermediate plane. The right member of (3a) maps the rectangular boundary in the z_1-plane onto the same real axis, since the sn-function is real on all four sides of the rectangle.

Note: This transformation is related to the transformation $\zeta' = -(k_0)^{1/2} \operatorname{sn}(z_0 ; k_0)$ by a Gauss transformation of elliptic functions (see, for example, reference [2]) and a translation of the coordinate axes. In the aerodynamic application mentioned above, the form (3) is found to be much simpler. The related form is discussed by Holzmüller [3].

*Received April 6, 1951. This paper is a chapter of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Michigan, December, 1950. The author is grateful to Dr. R. C. F. Bartels for his many helpful suggestions.

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The transformation (3) and equivalent combinations of functions have been used by previous investigators in the treatment of the linearized supersonic flow past a delta wing. (See, for example, reference [4].) We shall use it here as a basic transformation, from which more complicated transformations can be derived.

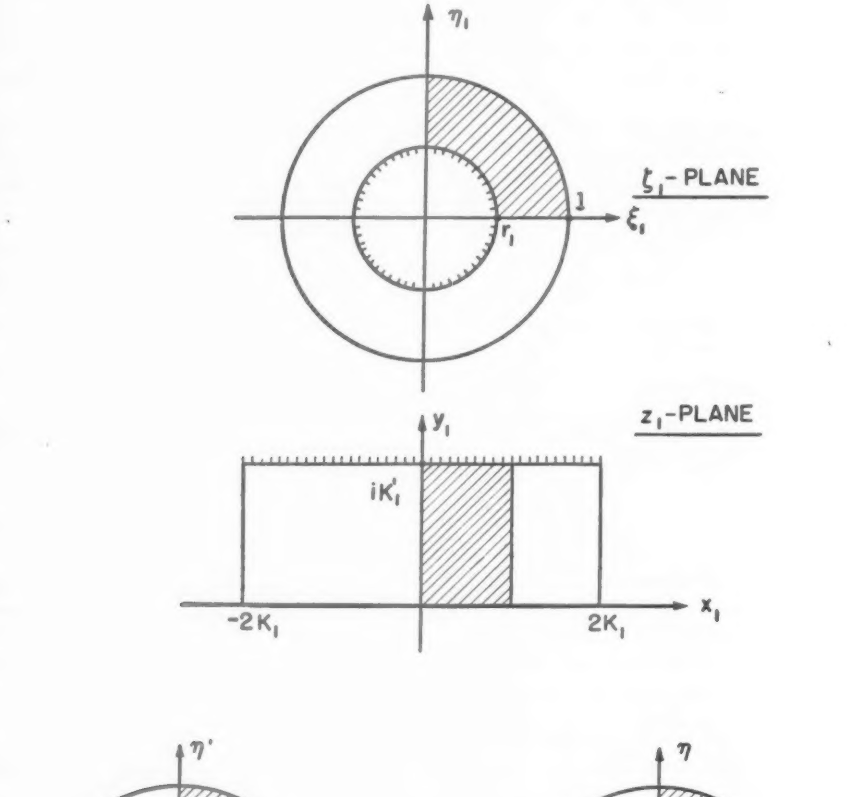

FIG. 1. Conformal Mapping of Domains Bounded by a Slot and the Unit Circle.

In equation (3) we may replace ζ' by $i\zeta$ and z_1 by $(z_1 + K_1)$. The resulting transformation collapses the circle $|\zeta_1| = r_1$ onto a vertical slot $\xi = 0$, $-(1 - k_1')/k_1 \leq \eta \leq (1 - k_1')/k_1$ (see Fig. 1). The transformation function is

$$\zeta = \text{cn}\,(z_1\,;k_1) + i\,\text{sn}\,(z_1\,;k_1). \tag{4}$$

An alternative form of equation (4) is the following:

$$\begin{gathered} i(1 - \zeta^2)/2\zeta = \text{sn}\,(z_1\,;k_1) \\ (-2K_1 < x_1 < 2K_1\,; \qquad 0 < y_1 < K_1') \end{gathered} \tag{4a}$$

The transformation (1), followed by either (3) or (4), leaves the included segments of both axes invariant as well as the unit circle. (See Fig. 1.)

3. Mapping of domains bounded by a cross and the unit circle. Because of the invariance of the axes, the horizontal slot may be opened into a circle with symmetric horizontal fins in the ζ_3-plane (see Fig. 2). We apply the inverse of transformations (1) and (2), but with a smaller modulus $k_3 < k_1$. That is, let

$$\zeta' = \frac{\text{cn}\,(z_3\,;k_3) + ik_3'\,\text{sn}\,(z_3\,;k_3)}{\text{dn}\,(z_3\,;k_3)}, \tag{5}$$

where

$$z_3 = (2K_3/\pi i)\log \zeta_3\,, \qquad (0 < \arg \zeta_3 < 2\pi). \tag{6}$$

The corresponding domain in the z_3-plane is a slotted rectangle (see Fig. 2).

The radius of the inner circle in the ζ_3-plane is r_3, given by the equation

$$r_3 = \exp\,(-\pi K_3'/2K_3). \tag{7}$$

An alternative form of equation (5) can be written, similar to equation (3a).

$$\begin{gathered} -(1 + \zeta'^2)/2\zeta' = \text{sn}\,(z_3 - K_3\,;k_3) \\ (-2K_3 < x_3 < 2K_3\,; \qquad 0 < y_3 < K_3') \end{gathered} \tag{5a}$$

If we combine equations (3a) and (5a), we may map directly from the z_1-plane to the z_3-plane. The resulting transformation may be written in four forms: (Related functions have been given by Kronsbein [5].)

$$\begin{gathered} \text{sn}\,(z_1 - K_1\,;k_1) = \text{sn}\,(z_3 - K_3\,;k_3) \\ \text{cn}\,(z_1 - K_1\,;k_1) = \text{cn}\,(z_3 - K_3\,;k_3) \\ \text{dn}\,(z_1 - K_1\,;k_1) = \pm[1 - k_1^2\,\text{sn}^2\,(z_3 - K_3\,;k_3)]^{1/2} \\ \pm[1 - k_3^2\,\text{sn}^2\,(z_1 - K_1\,;k_1)]^{1/2} = \text{dn}\,(z_3 - K_3\,;k_3) \\ (-2K_1 < x_1 < 2K_1\,; 0 < y_1 < K_1'\,; -2K_3 < x_3 < 2K_3\,; 0 < y_3 < K_3') \end{gathered} \tag{8}$$

If we argue in similar manner with equation (4a), we may construct a domain bounded internally by a circle with symmetric vertical fins in the ζ_2-plane. (See Fig. 2.)

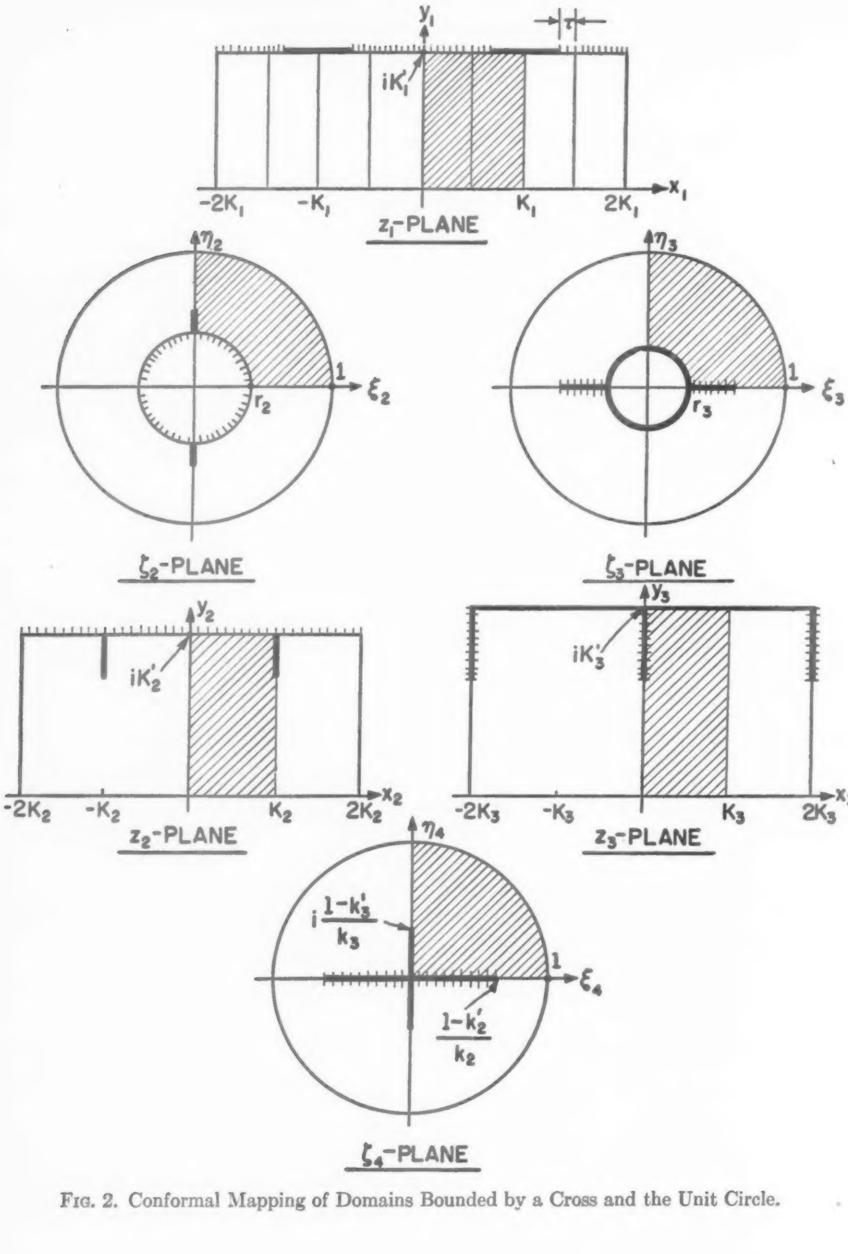

FIG. 2. Conformal Mapping of Domains Bounded by a Cross and the Unit Circle.

Again a new modulus $k_2 < k_1$ is used. The resulting transformation may be written in four forms:

$$\begin{aligned} \operatorname{sn}(z_1\,;k_1) &= \operatorname{sn}(z_2\,;k_2) \\ \operatorname{cn}(z_1\,;k_1) &= \operatorname{cn}(z_2\,;k_2) \\ \operatorname{dn}(z_1\,;k_1) &= \pm[1 - k_1^2 \operatorname{sn}^2(z_2\,;k_2)]^{1/2} \\ \pm[1 - k_2^2 \operatorname{sn}^2(z_1\,;k_1)]^{1/2} &= \operatorname{dn}(z_2\,;k_2) \end{aligned} \tag{9}$$

$$(-2K_1 < x_1 < 2K_1\,; 0 < y_1 < K_1'\,; -2K_2 < x_2 < 2K_2\,; 0 < y_2 < K_2')$$

where

$$z_2 = (2K_2/\pi i) \log \zeta_2\,, \qquad (0 < \arg \zeta_2 < 2\pi). \tag{10}$$

The corresponding domain in the z_2-plane is again a slotted rectangle. (See Fig. 2.) The radius of the inner circle in the ζ_2-plane is r_2, given by the equation

$$r_2 = \exp(-\pi K_2'/2K_2). \tag{11}$$

A transformation similar to equation (3) will collapse the circle $|\zeta_2| = r_2$ onto the real axis in the ζ_4-plane, leaving the vertical fins on the imaginary axis. (See Fig. 2.) The transformation equation is the following:

$$\zeta_4 = \frac{\operatorname{cn}(z_2\,;k_2) + ik_2' \operatorname{sn}(z_2\,;k_2)}{\operatorname{dn}(z_2\,;k_2)} \tag{12}$$

A transformation similar to equation (4) will collapse the circle $|\zeta_3| = r_3$ onto the imaginary axis in the ζ_4-plane, leaving the horizontal fins on the real axis. (See Fig. 2.) The transformation equation is the following:

$$\zeta_4 = \operatorname{cn}(z_3\,;k_3) + i \operatorname{sn}(z_3\,;k_3) \tag{13}$$

The two definitions of ζ_4 in equations (12) and (13) are equivalent if and only if the moduli are related by the equation

$$k_1' = k_2'k_3'\,. \tag{14}$$

[This statement can easily be proved in the form (17).] Then the inner boundary in the ζ_4-plane is a cross consisting of the horizontal slot, $-(1 - k_2')/k_2 \le \xi_4 \le (1 - k_2')/k_2$, $\eta_4 = 0$, and the vertical slot, $\xi_4 = 0$, $-(1 - k_3')/k_3 \le \eta_4 \le (1 - k_3')/k_3$.

By a series of steps we have mapped the circle $|\zeta_1| = r_1$ (Fig. 1) onto a cross, the unit circle remaining invariant (as well as the coordinate axes).

It is convenient to introduce a shape factor ϵ for the cross, defined by the relation

$$\epsilon = k_2'/(k_1')^{1/2} = (k_1')^{1/2}/k_3'\,. \tag{15}$$

ϵ has the range $(k_1')^{1/2} \le \epsilon \le (k_1')^{-1/2}$. Then the moduli k_2 and k_3 and the respective complementary moduli are given in terms of ϵ and k_1 by the equations

$$\begin{aligned} k_2 &= (1 - \epsilon^2 k_1')^{1/2}, & k_2' &= \epsilon(k_1')^{1/2}, \\ k_3 &= (1 - k_1'/\epsilon^2)^{1/2}, & k_3' &= (k_1')^{1/2}/\epsilon. \end{aligned} \tag{16}$$

If $\epsilon = 1$, then $k_2' = k_3' = (k_1')^{1/2}$ and the vertical fins have the same length as the horizontal fins. If $\epsilon = (k_1')^{-1/2}$, then $k_2 = 0$, $k_3 = k_1$, $r_2 = 0$, and equations (8) become an identity. Then the cross degenerates to a vertical slot. If $\epsilon = (k_1')^{1/2}$, then $k_2 = k_1$, $k_3 = 0$, $r_3 = 0$, and equations (9) become an identity. Then the cross degenerates to a horizontal slot. In general, if $\epsilon > 1$, the vertical fins are longer than the horizontal fins. If $\epsilon < 1$, the horizontal fins are longer. (Fig. 2 is drawn for $\epsilon < 1$.)

The combination of equations (9) and (12) or of equations (8) and (13) gives the resulting equation

$$\zeta_4^2 = \frac{\operatorname{cn}(z_1\,;k_1) + i\epsilon(k_1')^{1/2}\operatorname{sn}(z_1\,;k_1)}{\operatorname{cn}(z_1\,;k_1) - i\epsilon(k_1')^{1/2}\operatorname{sn}(z_1\,;k_1)}. \tag{17}$$

Although the function ζ_4^2 is an elliptic function, the function ζ_4 is not. It is doubly periodic, but it has branch points. In particular, there are branch points at the images of the point $\zeta_4 = 0$, the intersection of the cross. These points are defined by the equations

$$z_1 = \pm\left(\frac{1}{2}K_1 + \tau\right) + iK_1', \qquad \pm\left(\frac{3K_1}{2} - \tau\right) + iK_1'. \tag{18}$$

[These points are the zeros of the functions $\operatorname{dn}(z_2\,;k_2)$ and $\operatorname{dn}(z_3 - K_3\,;k_3)$. See Fig. 2.] If $\epsilon \leq 1$, $\tau \geq 0$. If $\epsilon \geq 1$, $\tau \leq 0$. If $\epsilon = (k_1')^{1/2}$, then $\tau = \frac{1}{2}K_1$ and the branch points coalesce to form simple zeros at the points $z_1 = \pm K_1 + iK_1'$, the zeros of the function (3). If $\epsilon = (k_1')^{-1/2}$, then $\tau = -\frac{1}{2}K_1$ and the branch points coalesce to form simple zeros at the points $z_1 = iK_1'$, $\pm 2K_1 + iK_1'$, as in the function (4).

Let us derive the mapping from the ζ_1-plane onto the ζ_4-plane by a distinctly different method. First, we square the variable ζ_1, making the two circles doubly covered, the inner one being of radius r_1^2. Let the appropriate modulus be $\bar{k}_1$, so that the corresponding complete elliptic integrals are related to the radius by the equation

$$r_1^2 = \exp(-\pi\bar{K}_1'/2\bar{K}_1). \tag{19}$$

Second, apply the transformation (3) with modulus $\bar{k}_1$ in the two-sheeted plane, preserving the doubly covered unit circle and creating a quadruply covered horizontal slot. Third, extract the square root. The final boundaries are the simply covered unit circle and a cross with four fins of equal length. The resulting transformation may be written as follows: (A related function has been given by Kronsbein [5].)

$$(\zeta_4')^2 = \frac{\operatorname{cn}([4\bar{K}_1/\pi i]\log\zeta_1\,;\bar{k}_1) + i\bar{k}_1'\operatorname{sn}([4\bar{K}_1/\pi i]\log\zeta_1\,;\bar{k}_1)}{\operatorname{dn}([4\bar{K}_1/\pi i]\log\zeta_1\,;\bar{k}_1)} \tag{20}$$

Comparison of equations (2) and (19) reveals that the moduli k_1 and $\bar{k}_1$ are related by a Landen transformation. (See, for example, reference [6].) If we apply the Landen transformation, whereby

$$\begin{aligned} \bar{k}_1 &= (1 - k_1')/(1 + k_1'), \qquad & \bar{k}_1' &= 2(k_1')^{1/2}/(1 + k_1'), \\ \bar{K}_1 &= \tfrac{1}{2}(1 + k_1')K_1, \qquad & \bar{K}_1' &= (1 + k_1')K_1', \end{aligned} \tag{21}$$

and

$$
\begin{aligned}
\operatorname{sn}([1+k_1']z_1\,;\bar{k}_1) &= (1+k_1')\frac{\operatorname{sn}(z_1\,;k_1)\operatorname{cn}(z_1\,;k_1)}{\operatorname{dn}(z_1\,;k_1)},\\
\operatorname{cn}([1+k_1']z_1\,;\bar{k}_1) &= \frac{1-(1+k_1')\operatorname{sn}^2(z_1\,;k_1)}{\operatorname{dn}(z_1\,;k_1)},\\
\operatorname{dn}([1+k_1']z_1\,;\bar{k}_1) &= \frac{1-(1-k_1')\operatorname{sn}^2(z_1\,;k_1)}{\operatorname{dn}(z_1\,;k_1)},
\end{aligned} \tag{22}
$$

then equation (20) may be written

$$(\zeta_4')^2 = \frac{\operatorname{cn}(z_1\,;k_1) + i(k_1')^{1/2}\operatorname{sn}(z_1\,;k_1)}{\operatorname{cn}(z_1\,;k_1) - i(k_1')^{1/2}\operatorname{sn}(z_1\,;k_1)}, \tag{23}$$

where z_1 is given as in (1).

We observe that equation (23) is indeed the mapping function for a cross of equal lengths, for it is a special case of equation (17), namely $\epsilon = 1$.

The cross of equal lengths can easily be mapped into the cross of unequal lengths in Fig. 2. We apply a linear fractional transformation which preserves the unit circle in the two-sheeted plane. If the transformation

$$\zeta_4^2 = \frac{(\zeta_4')^2 - (\epsilon-1)/(\epsilon+1)}{1-[(\epsilon-1)/(\epsilon+1)](\zeta_4')^2} \tag{24}$$

is combined with equation (23), then equation (17) is the result.

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EFFECT OF A RIGID ELLIPTIC DISK ON THE STRESS DISTRIBUTION IN AN ORTHOTROPIC PLATE*

By A. J. OWENS and C. B. SMITH, (*University of Florida*)

A thin orthotropic plate of uniform thickness will possess two perpendicular axes of symmetry in the plane of the plate. An infinite rectangular plate of this type containing a rigid elliptic disk with major and minor axes coinciding with the axes of symmetry is discussed. A uniform tension is assumed to act along two opposite edges of the plate and a mathematical analysis of the stress distribution is given. It is assumed the

*Received Feb. 21, 1951.

strains are small and remain within the limits of perfect elasticity. The solution obtained is applied to a plain-sawn Sitka spruce plate.

Choose as the origin of the coordinate system the center of the ellipse. The boundary conditions may be stated mathematically using the notation of A. E. H. Love[1] as

$$\left.\begin{aligned} X_x \mid_{x\to\pm\infty} &= S, & Y_y \mid_{y\to\pm\infty} &= 0, \\ X_y \mid_{x\to\pm\infty} &= 0, & X_y \mid_{y\to\pm\infty} &= 0, \end{aligned}\right\} \tag{1}$$

and $u(x, y) = v(x, y) = 0$ on the circumference of the disk given by the parametric equations $x = a \cos \theta$, $y = b \sin \theta$. The displacements in the x and y directions are $u(x, y)$ and $v(x, y)$, respectively. S is the uniform tension applied at two edges of the plate.

The components of stress and strain are connected by the following relations if the x and y axes are taken as the axes of elastic symmetry of the orthotropic plate:[2]

$$e_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E_x} X_x - \frac{\sigma_{yx}}{E_y} Y_y , \tag{2}$$

$$e_{yy} = \frac{\partial v}{\partial y} = -\frac{\sigma_{xy}}{E_x} X_x + \frac{1}{E_y} Y_y , \tag{3}$$

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{\mu_{xy}} X_y . \tag{4}$$

It is desirable to find a stress function $F(x, y)$[3] such that

$$X_x = \frac{\partial^2 F}{\partial y^2}, \qquad Y_y = \frac{\partial^2 F}{\partial x^2}, \qquad \text{and} \qquad X_y = -\frac{\partial^2 F}{\partial x\, \partial y}. \tag{5}$$

For the problem of a thin orthotropic plate in a state of plane stress F must satisfy the differential equation[4]

$$\frac{\partial^4 F}{\partial x^4} + 2K \frac{\partial^4 F}{\partial x^2\, \partial \eta^2} + \frac{\partial^4 F}{\partial \eta^4} = 0 \tag{6}$$

where

$$K = \tfrac{1}{2}(E_x E_y)^{1/2} \left(\frac{1}{\mu_{xy}} - \frac{2\sigma_{xy}}{E_x}\right), \tag{7}$$

$$\eta = \epsilon y, \qquad \text{and} \qquad \epsilon = (E_x/E_y)^{1/4}. \tag{8}$$

A suitable stress function is

$$\begin{aligned} F = R\Big\{ &\frac{A}{2\gamma_1^2}\left[\frac{1}{2}(Z_1 - W_1)^2 + \gamma_1^2 \ln (Z_1 + W_1)\right] \\ &+ \frac{B}{2\gamma_2^2}\left[\frac{1}{2}(Z_2 - W_2)^2 + \gamma_2^2 \ln (Z_2 + W_2)\right]\Big\} \\ &+ \frac{S\eta^2}{2\epsilon^2} \end{aligned} \tag{9}$$

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[1]A. E. H. Love, *A treatise on the mathematical theory of elasticity*, Dover Publications, New York, 1944.

[2]H. W. March, *Stress-strain relations in wood and plywood considered as orthotropic materials*, Forest Products Laboratory Rep. No. R1503, p. 2 (1944).

where the symbol $R\{\quad\}$ denotes the real part,

$$A = A_1 + iA_2, \qquad B = B_1 + iB_2, \tag{10}$$

$$Z_1 = x + i\alpha\eta, \qquad Z_2 = x + i\beta\eta, \qquad \eta = \epsilon y, \tag{11}$$

$$W_1 = (Z_1^2 - \gamma_1^2)^{1/2}, \qquad W_2 = (Z_2^2 - \gamma_2^2)^{1/2}, \tag{12}$$

$$\gamma_1^2 = a^2 - \alpha^2\epsilon^2 b^2, \qquad \gamma_2^2 = a^2 - \beta^2\epsilon^2 b^2, \tag{13}$$

$$\alpha = \{K + (K^2 - 1)^{1/2}\}^{1/2},$$

and

$$\beta = \{K - (K^2 - 1)^{1/2}\}^{1/2}.$$

Using the transformation

$$K = \cosh \phi,$$

$$\alpha = e^{\phi/2} \qquad \text{and} \qquad \beta = e^{-\phi/2}. \tag{14}$$

In order that the stresses may be single valued W_1 and W_2 are to be assigned values so that the inequalities

$$| Z_1 + W_1 | \geqq \gamma_1 \qquad \text{and} \qquad | Z_2 + W_2 | \geqq \gamma_2 \tag{15}$$

are satisfied.

It follows that

$$Y_y = \frac{\partial^2 F}{\partial x^2} = R\left\{\frac{-A}{W_1(Z_1 + W_1)} + \frac{-B}{W_2(Z_2 + W_2)}\right\}, \tag{16}$$

$$X_x = \epsilon^2 \frac{\partial^2 F}{\partial \eta^2} = R\left\{\frac{\alpha^2\epsilon^2 A}{W_1(Z_1 + W_1)} + \frac{\beta^2\epsilon^2 B}{W_2(Z_2 + W_2)}\right\} + S, \tag{17}$$

and

$$X_y = -\epsilon \frac{\partial^2 F}{\partial x\, \partial \eta} = R\left\{\frac{i\alpha\epsilon A}{W_1(Z_1 + W_1)} + \frac{i\beta\epsilon B}{W_2(Z_2 + W_2)}\right\}. \tag{18}$$

It can be shown that the exterior boundary conditions are satisfied so long as A and B are finite.

Substitution of the stresses given by (16) and (17) into (2) and integration gives

$$u = R\left\{A\left[\frac{\alpha^2\epsilon^2}{E_x} + \frac{\sigma_{yx}}{E_y}\right]\left[\frac{-1}{Z_1 + W_1}\right] + B\left[\frac{\beta^2\epsilon^2}{E_x} + \frac{\sigma_{xy}}{E_y}\right]\left[\frac{-1}{Z_2 + W_2}\right]\right\} + \frac{Sx}{E_x} + u_0(y)$$

where $u_0(y)$ is an arbitrary function which can be shown to be identically zero. A similar expression for v may be found by substituting the stresses in (3) and integrating. From the conditions $u = v = 0$ on the boundary of the disk four equations are obtained which uniquely determine A_1, A_2, B_1 and B_2.

[3]G. B. Airy, Rep. Brit. Assoc. Adv. Sci., 1862.

[4]C. Bassel Smith, Q. App. Math. 6, 452-456 (1949).

It follows that

$$A_2 = B_2 = 0;$$

$$A_1 = \frac{(a + \alpha\epsilon b)[Sa\alpha(1 + \epsilon^2\beta^2\sigma_{yx}) + Sb\epsilon\sigma_{yx}(\epsilon^2\beta^2 + \sigma_{xy})]}{\alpha(\epsilon^2\alpha^2 + \sigma_{xy})(1 + \epsilon^2\beta^2\sigma_{yx}) - \beta(1 + \epsilon^2\alpha^2\sigma_{yx})(\epsilon^2\beta^2 + \sigma_{xy})}; \tag{19}$$

and

$$B_1 = -\frac{(a + \beta\epsilon b)[Sa\beta(1 + \epsilon^2\alpha^2\sigma_{yx}) + Sb\epsilon\sigma_{yx}(\epsilon^2\alpha^2 + \sigma_{xy})]}{\alpha(\epsilon^2\alpha^2 + \sigma_{xy})(1 + \epsilon^2\beta^2\sigma_{yx}) - \beta(1 + \epsilon^2\alpha^2\sigma_{yx})(\epsilon^2\beta^2 + \sigma_{xy})}. \tag{20}$$

The stress function is now determined and it can be shown that it contains as a special case the stress function for an isotropic plate. Let $a = b$ and set $\epsilon = 1$. Then from (6) it is seen that $E_x = E_y$. This is a necessary condition but not a sufficient condition that the plate in question be an isotropic plate. For the isotropic case, it is now sufficient that $\phi \to 0$; since by (14) $K \to 1$ and (6) reduces to the biharmonic equation for the isotropic case. Parts of the stress function become indeterminate but they may be evaluated by successive applications of L'Hospitals rule.

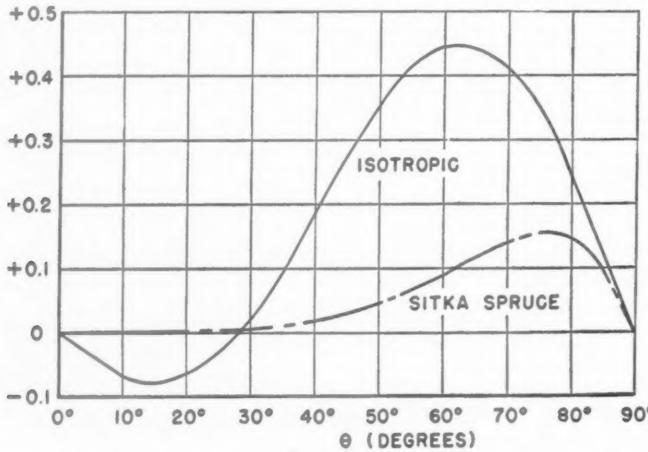

FIG. 1. Variation of the shear stress component X_y at points along the boundary of a rigid circular disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.—Ordinates: Ratio of shear stress component X_y at points on the boundary of a rigid circular disk to S the normal stress at infinity.

The stress function for a plate in a state of plane strain was also determined and it was shown that it also contained as a special case the stress function for an isotropic plate. It is interesting to note that using a different method Professor I. S. Sokolnikoff[5] obtained the same stress function for an isotropic plate.

Consider now a large plain-sawn rectangular plate of Sitka spruce containing a rigid elliptic disk at its center. It is possible to apply the results to a finite plate since the

[5] I. S. Sokolnikoff, *Mathematical theory of elasticity* (mimeographed lecture notes, Brown University, 1941) Chap. V.

stress concentration is localized near the rigid disk. The x- and y- axes are parallel and perpendicular to the grain, respectively.

Shear stresses are probably most important in producing failure in a wooden plate. The shear stress given by (18) evaluated on the boundary of the rigid disk is

$$X_y \Big|_{0 \leq \theta \leq \pi/2} = \frac{\alpha\epsilon A_1 \sin\theta\cos\theta}{\alpha^2\epsilon^2 b^2\cos^2\theta + a^2\sin^2\theta} + \frac{\beta\epsilon B_1 \sin\theta\cos\theta}{\beta^2\epsilon^2 b^2\cos^2\theta + a^2\sin^2\theta}. \tag{21}$$

In Figure 1 the curve for Sitka spruce was plotted utilizing formulas (19), (20) and (21). The isotropic curve was computed for $\sigma = 0.3$.

Throughout the Sitka spruce plate containing a rigid circular disk $| Y_y |$ is less than $0.03S$. The maximum value of X_x is $1.23S$ at the point $(a, 0)$.

From these results, it is probable that, for S sufficiently large, failure in the plate will occur approximately along the lines $y = \pm a \sin 77\tfrac{1}{2}° = \pm 0.976a$ with the crack beginning at the edge of the disk.

BOOK REVIEWS

The mathematical theory of plasticity. By R. Hill. Oxford at the Clarendon Press, 1950. ix + 356 pp. $7.00.

Time, temperature, Bauschinger, hysteresis, and size effects are all explicitly ruled out and the major but not exclusive emphasis is on ideal plasticity. However, a wide range of topics is discussed in this interesting and invaluable treatise. Mathematical proofs, experimental evidence and the practical evaluation of theory are kept in excellent balance from the study of stress-strain relations, variational principles, and the questions of uniqueness, to the solving of practical metal forming problems. A few relatively simple elastic-plastic solutions are included for prismatic bars, and for thick cylindrical tubes and spherical shells. However, most of the chapters deal with plastic-rigid techniques and solutions developed by the author and E. H. Lee. An extensive discussion is given of the slip line fields for plane problems. The necessity for complete solutions is stated strongly and repeated warning is given against the error of thinking in terms of static determinacy only and not considering velocity conditions as well. A number of interesting and truly amazing solutions are given in detail for which the configuration remains geometrically similar as the plastic deformation proceeds. Among the miscellaneous subjects covered are: machining, hardness tests, notched bars, normal and oblique necking, earing and anisotropy. Tensor notation is used throughout without apology.

As is entirely proper for a book on the mathematical theory of plasticity, the physics of metals is covered only by reference to treatises on the subject. A very brief Appendix on suffix notation, the summation convention, and hyperbolic differential equations is considered sufficient for the reader who is assumed to be familiar with the elementary theory of elasticity.

The reviewer regretted to detect an apparent desire on the part of the author to have written a truly definitive text at a stage in the development of plasticity when too many basically new facts are being discovered. As the author himself is in the forefront of many of these developments he could have done a greater service by indicating more clearly the needed fundamentally new directions rather than glossing over our present shortcomings and often embarrassing lack of knowledge. He then might not have allowed himself to get so worried by Southwell and Allen's elastic-plastic solution for a notched bar that

he unjustifiably loses faith in his rigid plastic solutions. Also, there would be no need to keep a distinction between the true yield function and the plastic potential in the derivation of general stress-strain relations, nor to place so much faith in the path independence of the dissipated work.

D. C. DRUCKER

Response of physical systems. By John D. Trimmer. John Wiley & Sons, Inc., New York and Chapman & Hall, Limited, London, 1950. ix + 268 pp. $5.00.

This interesting book was written from material used in a course in "Instrumentation" primarily for physicists and engineers. The book is interesting because it manages to cover in concise form, on a level suitable for seniors, the essentials of system response to sinusoidal and pulse driving. Both mechanical and electrical systems are dealt with and system stability is treated in sufficient detail to give a good idea of the problems. Measuring instruments are discussed as physical systems and measurements as the response of such systems together with measurement errors. The Chapter headings are: A Pattern for Systems, Physical Systems, First Order Systems, Second Order Systems, Sinusoidal Forcing of Linear Systems, Higher Order Systems, Measuring Instruments, Feedback Systems, Parametric Forcing, Distributed Systems, Nonlinear Systems. Appendices: The Laplace Transform in the Study of Linear Systems, Problems and Review Questions.

R. TRUELL

Classical mechanics. By H. C. Corben and Philip Stehle. John Wiley & Sons, Inc., New York and Chapman & Hall, Ltd., London, 1950. xvii + 388 pp. $6.50.

As the authors point out in the Preface "the parts of classical mechanics which are of present-day interest to the physicist are not those which were of paramount interest in the nineteenth century." The authors therefore set out to write a modern book which would reflect this change in emphasis in an old field. In this reviewer's opinion, they have succeeded to a remarkable extent in producing such a text. The space available for this review does not permit a detailed discussion of the contents, and the following list of chapter headings (with some significant section headings added in parentheses) must suffice. Kinematics of particles. The laws of motion. Conservative systems with one degree of freedom. Miscellaneous theorems on systems of particles Lagrange's equations of motion (Charged particle in an electromagnetic field). Applications of Lagrange's equations (Rutherford scattering. The cylindrical magnetron). Linear vector spaces. Small oscillations of conservative systems (Application to molecules. The molecule X_3Y). Rigid bodies. Hamiltonian theory. Contact transformations. Contact transformations which simplify the equations of motion. Poisson brackets (Poisson brackets in quantum mechanics). Infinitesimal contact transformations. Further development of transformation theory. Miscellaneous generalizations and analogies (Analogy with thermodynamics). Introduction to special relativity theory. The motion of particles in high energy accelerators.

W. PRAGER

Methods of mathematical physics. By Harold Jeffreys and Bertha Swirles Jeffreys. University Press, Cambridge, 1950. vii + 708 pp. $15.00.

The first edition of this treatise appeared in 1946 [see this Quarterly **5**, 366(1947)]. The fact that a second edition should have become necessary so soon clearly indicates that the work has become a standard reference for mathematical physicists and applied mathematicians. While the material has been

rearranged in some chapters and minor changes have been made throughout the book, the major changes are as follows. A section on block matrices has been added, the chapter on multiple integrals has been thoroughly revised, relaxation methods have been treated in somewhat greater detail, the presentation of the theory of inverse functions has been revised, and the treatment of multipole radiation has been extended. Also, some new problems have been added.

W. PRAGER

Foundations of aerodynamics. By A. M. Kuethe and J. D. Schetzer. John Wiley & Sons, Inc., New York and Chapman & Hall, Ltd., London, 1950. ix + 374 pp. $5.75.

This is a textbook for a general course in aerodynamics, intended for graduate and senior-undergraduate students whose training has been in engineering, mathematics, or physics, but who need not previously have studied aerodynamics. There are three major parts of the volume, devoted substantially to incompressible-inviscid, compressible-inviscid, and incompressible-viscous flows. It is a major attraction of the textbook, however, that real gases are discussed in the opening chapter. Here, in addition, the plan of the whole book is laid out for the student and the admirable objective of his study is stated: "to provide a background of sound concepts for finding approximate solutions of problems in the flow of a compressible, viscous, inhomogeneous gas."

The chapters based on incompressible-flow theory include, among other matters, classical two-dimensional thin-airfoil theory and Prandtl wing theory. The compressible-fluid chapters include channel flow, shock and expansion waves, Prandtl-Glauert rule, and linearized supersonic airfoil theory. Except for a descriptive section, the last-mentioned is confined to plane flow and the equivalent sweptback infinite wings.

The stress terms in the Navier-Stokes equations are derived in an Appendix, in lucid style. Two chapters concern mostly laminar boundary-layer flow; two more are devoted to turbulence, and another to boundary-layer transition, including descriptions of boundary-layer instabilities of two types. These chapters seem particularly valuable, and are probably unique in the field of fluid-mechanics textbooks.

The closing chapters present material on viscous-compressible flow and some descriptive information on wing properties, such as transonic flow and stalling. The viscous-compressible paragraphs are soundly physical, concerning especially energy balance and aerodynamic heating. There is also some information on interactions between shock wave and boundary layer, and a regrettably short paragraph on "the thickness of a shock wave."

There are problems for the first 13 of the 18 chapters, which seem, to the reviewer, to demand rather little originality from the student. There are gratifying exceptions to this generalization, however.

Throughout, the authors have emphasized the physical picture that their equations describe. Moreover, the reader is frequently reminded of the confirmation, and occasional conflict, of experimental results. Thus, although it will undoubtedly teach most seniors and young graduate students a good deal of applied mathematics, it is really not a book in that field, but one in the field of physics, as it should be.

In view of the purpose and the scope of the volume, it is probably not justifiable to complain about sins of omission; rather, the authors should be commended on their skill in having selected their topics well. It may suffice to remark that no single volume of 374 pages can be adequate for full-dress presentations of theoretical aerodynamics at the graduate level. In its own field—which is one that should be vigorously encouraged—this is certainly the best textbook, and a badly-needed one.

W. R. SEARS

Theory of probability. By M. E. Munroe. First Edition. McGraw-Hill Book Company, New York, Toronto, London, 1951. viii + 211 pp. $4.50.

This is an elementary textbook designed to introduce students who have had a first year course in calculus to the theory of probability. The author makes frequent references to Cramér's and Uspensky's

books, and the general purpose of the book seems to be to provide a book more accessible to less advanced students than these books are. As a matter of fact, the present book is more in the spirit of probability theory than is Uspensky's book. Even a version of the strong law of large numbers is discussed. However, strong convergence is defined as convergence in the space of infinite sequences. This is rather sophisticated for students with a year of calculus and an alternate limit definition might have been more intelligible. The book has many illustrative problems and exercises for the student.

J. WOLFOWITZ

Vector and tensor analysis. By Harry Lass. McGraw-Hill Book Company, Inc., New York, Toronto and London, 1950. xi + 347 pp. $4.50.

This book contains a thorough treatment of vector analysis, with extensive applications. It also contains a short but good introduction to tensors. There is a large number of examples and problems. The style is clear and concise, and there is a sharp segregation between general theory and the many applications. This renders the book suitable for the standard undergraduate course in vector analysis, as well as for more advanced work dealing with an introducton to mathematical physics.

There are nine chapters. Chapters I, II and IV deal with the algebra, differential calculus and integral calculus of vectors. These chapters, comprising about one quarter of the book, contain the theory appearing in the standard undergraduate course in vector analysis. Chapters III, V, VI and VII, which comprise about one half of the book, deal respectively with differential geometry, electricity, mechanics, and hydrodynamics and elasticity. Of course, in a book of this nature the treatments of these topics can be an introduction only, but the treatment is systematic, and particularly in the case of electricity is rather extensive. Chapters VIII and IX, which comprise the remaining one quarter of the book, contain a brief introduction to the theory of tensors, as well as brief applications of tensors to geometry, mechanics and elasticity.

G. E. HAY

Elasticity. Proceedings of the Third Symposium in Applied Mathematics of the American Mathematical Society, Volume III. McGraw-Hill Book Company, Inc., New York, Toronto, London, 1950. v + 233 pp. $6.00.

This volume contains the papers presented at the Third Symposium on Applied Mathematics of the American Mathematical Society, held at the University of Michigan on June 14-16, 1949. Seventeen papers are included, two of these being presented in the form of abstracts.

The subject matter of thirteen of the papers is elasticity, that of the remainder, plasticity. The papers in elasticity include studies in the plane problems of anisotropic materials, finite deflections of axisymmetric shells and asymptotic integration in shell theory, the bending of plates, general theory of finite elastic deformation, the beam of circular cross-section under concentrated loading parallel to its axis, approximate determination of the torsional rigidity of beams, torsion of the axially symmetric shaft, and the boundary layer edge effect in elastic plates. The plasticity contributions are on the stress analysis of indeterminate elastic-plastic structures, the use of characteristics in obtaining graphic solutions of plane problems, and the solution of problems in plane flow with stress discontinuities.

WILLIAM H. PELL

SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu amd u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\sqrt{\ }$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u\, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t\, (a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354-372 (1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

The Eighth International Congress for Applied Mechanics

will be held at the University of Istanbul in Istanbul, Turkey, from August 20 to 28, 1952.

Membership. Membership in the Congress will be open to all qualified persons, whether they are able to be present in person or not. For regular members of the Congress the fee is $10: these persons will be entitled to participate in all the scientific and social features of the Congress and may, if they wish, contribute papers. Members of families, accompanying Congress members and not participating in the scientific meetings, may become associate members for a fee of $5: they will be entitled to all the privileges of membership.

Contribution of papers. Each member of the Congress may present at most two papers, and the time allotted for each paper will not exceed fifteen minutes. Facilities for projection of formulae and graphs will be available in order to save time; members wishing to make use of these should prepare slides beforehand not exceeding 8 x 8 cm (3 x 3″). Abstracts for contributed papers should not exceed 400 words in length and must be submitted on blanks which may be secured from the Secretary of the Congress. Abstracts must be in the hands of the Organizing Committee not later than June 1st, 1952. The Organizing Committee will endeavour to accept papers arriving after this date but will be under no obligation to do so. Members of the Congress who will not be able to attend in person may send only one paper which must be read at the Congress by a member of the Congress designated by the author. The official languages of the Congress are English, French, German, and Italian.

The Sections for the presentation of short contributed papers will be as follows:

SECTION I: Elasticity—Plasticity—Rheology

SECTION II: Fluid Mechanics (Aerodynamics—Hydrodynamics)

SECTION III: Mechanics of Solids (Ballistics—Vibrations—Friction—Lubrication)

SECTION IV: Statistical Mechanics—Thermodynamics—Heat Transfer

SECTION V: Mathematics of Physics and Mechanics—Methods of Computation

The Proceedings of the Congress will contain in extenso the speeches, lectures, major addresses and short contributed papers read at the Congress. If any of these are not available, the abstracts will be printed in their place. Members who wish to receive the Proceedings of the Congress will be required to pay a fee which will be announced at a later date.

Five minutes will be allowed for the free discussion of the contents of each paper read at the Congress. The discussions will not be printed in the Proceedings.

Major addresses and lectures. The Organizing Committee intends to invite a few outstanding persons to deliver stated addresses. There will also be a lecture by a specialist in each section of the Congress, mainly in order to present research in fields in which important advances have been made in the recent years.

Entertainment. There will be many interesting entertainment features, amongst them a reception and a banquet which all members of the Congress are entitled to attend. Excursions by boat or bus are also being considered. Finally, members will be given the opportunity of visiting the main historical monuments, and architectural and natural beauties of Istanbul, under the guidance of competent persons. In connection with the Congress, Messrs. Henry Van der Zee and Co. of New York and Istanbul, acting as agents for the American Express Co. Inc. of New York, will provide for organized tours to Troy, Bergama, Ephesus, Bursa, Izmir, and Ankara.

Travel. The Organizing Committee has designated as the official travel agency of the Congress the American Express Company, Inc. of New York, from which all information concerning travel and accommodation can be obtained.

The Organizing Committee is providing for a 25% reduction on the one way fare and a 50% reduction on the round trip fare on all boats belonging to the Turkish State Lines, operating in the Mediterranean, and on trains of the Turkish State Railways. It is also expected that transportation by sea from the United States will be made available for approximately $450 for the round trip.

Accommodation. First class hotel rooms are available for $3.50 upwards, whilst second class rooms are from $2 to $3.50. Free accommodation will be provided in school dormitories.

Meals. Excellent meals in first class restaurants cost from $1.50 upwards. Also meals will be provided by the University cafeteria for approximately $0.65 upwards.

Information. All requests for information concerning the Congress should be addressed to:

The Eighth International Congress for Applied Mechanics, P. O. Box 245, Istanbul (Turkey)

Chairman of the Organizing Committee:
Kerim Erim.

Secretariat of the Congress:
Cahit Arf (General Secretary),
Giacomo Saban (Acting Secretary).

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